Solutions Sheet 3

Sheaves & Schemes

Note: If you have never seen the concept of direct limit, please have a look at the additional exercise A on the last page.

1. *Inverse Image Sheaf.* Let $f : X \to Y$ be a continuous map of topological spaces. For a sheaf $\mathscr{G}$ of abelian groups on Y we define the *inverse image* sheaf $f^{-1}\mathscr{G}$ on X to be the sheaf associated to the presheaf $U \mapsto \varinjlim_{V \supset f(U)} \mathscr{G}(V)$, where U is any open set of X and the direct limit (see exercise A) is taken over all open subsets V of Y containing $f(U)$. Prove that for every sheaf $\mathscr{F}$ on X there is a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$ and for any sheaf $\mathscr{G}$ on Y there is a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Prove that this induces a natural bijection of sets

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$

for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y. One says that f^{-1} and f_* are adjoint functors.

Solution: Let $\mathscr{F}$ be a sheaf on X. Denote $\mathscr{F}'$ for the presheaf $U \mapsto \lim_{V \supset f(U)}(f_*\mathscr{F})(V)$. For every $V \supset f(U)$ we have that $f^{-1}(V) \supset f^{-1}f(U) \supset U$ and since by definition $(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$ we conclude that there is a restriction map $(f_*\mathscr{F})(V) \to \mathscr{F}(U)$. Hence we get a morphism $\mathscr{F}' \to \mathscr{F}$ of presheaves. Since $f^{-1}f_*\mathscr{F}$ is the sheaf associated to $\mathscr{F}'$ and by the universal property of the associated sheaf we get a morphism of sheaves $f^{-1}f_*\mathscr{F} \to \mathscr{F}$.

Now let $\mathscr{G}$ be a sheaf on Y. Denote by $\mathscr{G}'$ the presheaf $U' \mapsto \lim_{V \supset f(U')}\mathscr{G}(V)$ on X. We have $U \supset ff^{-1}(U)$ and thus we get a map $\mathscr{G}(U) \to \mathscr{G}'(f^{-1}(U))$. Composing this with the sheafification map $\mathscr{G}'(f^{-1}(U) \to (f^{-1}\mathscr{G})(f^{-1}(U))$ we get a morphism of sheaves $\mathscr{G} \to f_*f^{-1}\mathscr{G}$.

Let $\varphi : f^{-1}\mathscr{G} \to \mathscr{F}$ be a morphism of sheaves on X. Using the previous construction we get a morphism $\mathscr{G} \to f_*f^{-1}\mathscr{G} \to f_*\mathscr{F}$. Conversely, every morphism $\psi : \mathscr{G} \to f_*\mathscr{F}$ of sheaves on X induces a morphism $f^{-1}\mathscr{G} \to f^{-1}f_*\mathscr{F} \to \mathscr{F}$. To see that these two constructions are inverse to each other, either trace through the individual maps and note that they are only restriction maps and inclusions to the direct limit, or look at the map induced on the stalks.

2. Let X be a topological space and let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves of abelian groups on X.

   (a) Prove that the induced map $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for every open subset $U \subset X$ if and only if the map on the stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective for every point $x \in X$.

(b) Show that this is not true for surjectivity: Let X be the topological space $X := \mathbb{C} \setminus \{0\}$ with standard topology, let $\mathscr{F} = \mathscr{G}$ be the sheaf of nowhere-zero continuous complex-valued functions and let $\varphi : \mathscr{F} \to \mathscr{G}$ be the morphism that sends a function f to f^2. Prove that for every point $x \in X$ the induced morphism on the stalks φ_x is surjective, but on global sections $\varphi(X)$ is not surjective.

Solution:

(a) Suppose $\varphi(U)$ is injective for every open subset $U \subset X$. Let $x \in X$ be a point. An element in the stalk $\mathscr{F}_x$ is an equivalence class of pairs (V, s_V) for all open neighbourhoods $V \subset X$ of x and $s_V \in \mathscr{F}(V)$. Assume that $\varphi_x([V, s_V]) = 0$. Then there exists an open neighbourhood $W \subset X$ of x such that $\varphi_x([V, s_V]) = [W, 0]$. But then $\varphi(W)(s_W) = 0$ and so $s_W = 0$. We conclude that $[V, s_V] = 0$.

Conversely suppose φ_x is injective for all points $x \in X$. Let $U \subset X$ be an open subset and let $s \in \mathscr{F}(U)$ be a section. Assume that $\varphi(U)(s) = 0$. Then for any point $x \in U$ we have $\varphi_x([U, s]) = 0$, hence $s|_V = 0$ for some open neighbourhood $V \subset U$ of x by injectivity of φ_x. Doing this for all points $x \in U$ we conclude that s is zero on an open cover of U. Since $\mathscr{F}$ is a sheaf we conclude that $s = 0$.

(b) On global sections this morphism is not surjective, because the function $z \mapsto z$ does not have a preimage. To see this, note that any root function $z \mapsto \sqrt{z}$ is not continuous on all of $\mathbb{C} \setminus \{0\}$. However, there is a preimage of every element in the stalk. To see this let $[U, f]$ be an element of $\mathscr{F}_x$ for a point $x \in X$, where $U \subset X$ is a neighbourhood of x and f is a continuous function on U. Let $V \subset X$ be a small (simply connected) open neighbourhood of $f(x)$, such that there exists a continuous square root function on V. Then $(f^{-1}(V), \sqrt{f})$ is an element of $\mathscr{F}_x$ and $\varphi_x([f^{-1}(V), \sqrt{f}]) = [f^{-1}(V), f] = [U, f]$.

3. Let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. Prove that if f is a homeomorphism and $f^\sharp$ is an isomorphism, then $(f, f^\sharp)$ is an isomorphism.

Solution: We need to construct an inverse morphism $(g, g^\sharp) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of locally ringed spaces. Since f is a homeomorphism, we can define $g := f^{-1}$. On sheaves however, the definition is not so clear, since $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$. For every open subset $U \subset X$ we define the homomorphism

$$g^\sharp(U) : \mathcal{O}_X(U) \to g_*\mathcal{O}_Y(U) = \mathcal{O}_Y(g^{-1}(U)) = \mathcal{O}_Y(f(U))$$

as

$$f^{\sharp -1}(f(U)) : \mathcal{O}_X(f^{-1}f(U)) = \mathcal{O}_X(U) \to \mathcal{O}_Y(f(U)).$$

This is well-defined and one sees that it provides an inverse of $(f, f^\sharp)$ as morphism of locally ringed spaces.

4. Let A be a ring and set $X := \mathrm{Spec}(A)$. Let $f \in A$ and let $U_f \subset X$ be the open complement of $V((f))$.

   (a) Show that the locally ringed space $(U_f, \mathcal{O}|_{U_f})$ is isomorphic to $\mathrm{Spec}(A_f)$.

   (b) For another element $g \in A$ describe the restriction map $\mathcal{O}(U_f) \to \mathcal{O}(U_{fg})$ in terms of a ring homomorphism $A_f \to A_{fg}$.

Solution:

   (a) Denote $Y := \mathrm{Spec}(A_f)$. The prime ideals of A which do not contain (f) are in one-to-one correspondence with prime ideals in A_f. This correspondence is given as the pullback of the localisation map $\varphi : A \to A_f$. This pullback preserves inclusions and thus we conclude that Y and U_f are homeomorphic as topological spaces via φ^*. Furthermore, for every point $\mathfrak{p} \in U_f \cong Y$ we have $(\mathcal{O}|_{U_f})_{\mathfrak{p}} = A_{\mathfrak{p}} \cong (A_f)_{\varphi(\mathfrak{p})} = \mathcal{O}_{Y,\mathfrak{p}}$ and the isomorphism is given by $\varphi_{\mathfrak{p}}$. Since a morphism of sheaves is an isomorphism if and only if it is an isomorphism on the stalks, we conclude that $\mathcal{O}|_{U_f} \cong \mathcal{O}_Y$ as sheaves. We conclude that $\mathrm{Spec}(A_f) \cong (U_f, \mathcal{O}|_{U_f})$ as locally ringed spaces via the isomorphism φ^*.

   (b) The restriction is just the localisation map $A_f \to (A_f)_g \cong A_{fg}$. To see this, recall the proof of the isomorphism $A_f \cong \mathcal{O}(U_f)$. There we have proved that the map sending an element $a/f^n \in A_f$ to the section taking a point $\mathfrak{p} \in \mathcal{O}(U_f)$ to $a/f^n \in A_{\mathfrak{p}}$ is an isomorphism. We see that we have a commutative diagram:

$$\begin{array}{ccc} A_f & \xrightarrow{\cong} & \mathcal{O}(U_f) \\ \downarrow & & \downarrow \\ A_{fg} & \xrightarrow{\cong} & \mathcal{O}(U_{fg}) \end{array}$$

   where the left vertical map is the localisation map and the right vertical map is the restriction.

5. Consider $S_1 := \mathrm{Spec}(\mathbb{Q}[X,Y]/(XY))$ and $S_2 := \mathrm{Spec}(\mathbb{Q}[X,Y]/(X^2+Y^2))$.

   (a) Compute $\mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(\mathbb{Q}), S_i)$ for $i = 1, 2$.

   (b) Deduce that S_1 and S_2 are not isomorphic schemes.

   (c) Let $S_1' := \mathrm{Spec}(\mathbb{Q}(i)[X,Y]/(XY))$ and $S_2' := \mathrm{Spec}(\mathbb{Q}(i)[X,Y]/(X^2+Y^2))$. Prove that $S_1' \cong S_2'$ as schemes.

Solution:

   (a) We have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(\mathbb{Q}), S_1) &\cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Q}[X,Y]/(XY), \mathbb{Q}) \\ &= \{(a,b) \in \mathbb{Q} \mid ab = 0\} = (\mathbb{Q} \times \{0\}) \cup (\{0\} \times \mathbb{Q}) \end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(\mathbb{Q}),S_2)&\cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Q}[X,Y]/(X^2+Y^2),\mathbb{Q})\\ &=\big\{(a,b)\in\mathbb{Q}\;\big|\;a^2+b^2=0\big\}=\{(0,0)\}\end{aligned}$$

(b) Since these two Hom-sets are not isomorphic (different cardinality), the schemes S_1 and S_2 can not be isomorphic.

(c) To give an isomorphism of schemes $S_2' \to S_1'$ it is enough to give an isomorphism of rings $\mathbb{Q}(i)[X,Y]/(XY) \to \mathbb{Q}(i)[X,Y]/(X^2+Y^2)$. We define this ring homomorphism by $X \mapsto X+iY$ and $Y \mapsto X-iY$. Indeed $XY \mapsto X^2+Y^2$ and so the homomorphism is well-defined. Furthermore the homomorphism defined by $\frac{1}{2}(X+Y) \leftarrow\!\shortmid X$ and $\frac{i}{2}(Y-X) \leftarrow\!\shortmid Y$ provides a both-sided inverse, hence it is an isomorphism.

6. Let X be a scheme. For any point $x \in X$ we define the Zariski tangent space T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k and let $k[\varepsilon]/(\varepsilon^2)$ be the *ring of dual numbers* over k. Show that to give a morphism of schemes over k of $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ to X is equivalent to giving a point $x \in X$ such that $k(x)=k$ and an element of T_x.

   Solution: Let $\varphi : \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)) \to X$ be a k-morphism. The ring $A := k[\varepsilon]/(\varepsilon^2)$ has only one prime ideal (ε), hence $\mathrm{Spec}(A)$ is only one point, say $\mathfrak{o}$. Thus we get a point $x := \varphi(\mathfrak{o}) \in X$. The morphism φ induces a ring homomorphism $\mathcal{O}_{X,x} \to A$ with $\varphi(\mathfrak{m}_x) \subset (\varepsilon)$. We can thus define a k-linear map $\mathfrak{m}_x \to A \to k$ given by $a \mapsto \frac{a}{\varepsilon} \mapsto \frac{a}{\varepsilon} \mod (\varepsilon)$, whose kernel is $\mathfrak{m}_x^2$. Thus we have a map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k$, i.e. a tangent vector.

   Conversely, suppose that we are given a point $x \in X$ and a tangent vector $f \in T_x$. Every element in the stalk $\mathcal{O}_{X,x}$ is uniquely given by a sum $a+b$ with $a \in k$ and $b \in \mathfrak{m}_x$. We define $\varphi(a+b) := a + f(b)\varepsilon \in A$. This defines a ring homomorphism $\mathcal{O}_{X,x} \to A$. This gives us a scheme homomorphism $\mathrm{Spec}(A) \to X$ by sending the point to x and on the sheaves we have $\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to A$ for $x \in U$ and $\mathcal{O}_X(U) \to 0$ if $x \notin U$.

7. *The Proj-construction.* Let $S = \bigoplus_{d\geqslant 0} S_d$ be a graded ring. Denote $S_+ = \bigoplus_{d>0} S_d$. We define the set $\mathrm{Proj}(S)$ as the set of all homogeneous prime ideals $\mathfrak{p} \subset S$ which do not contain all of S_+. For a homogeneous ideal $\mathfrak{a} \subset S$ we define the subset $V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Proj}(S) \mid \mathfrak{p} \supset \mathfrak{a}\}$. These sets form the closed sets of a topology on $\mathrm{Proj}(S)$. We define a sheaf of rings $\mathcal{O}$ on $\mathrm{Proj}(S)$ as follows: For each $\mathfrak{p} \in \mathrm{Proj}(S)$ we consider the ring $S_{(\mathfrak{p})}$ of elements of degree zero in the localized ring $T^{-1}S$, where T is the multiplicative system of all homogeneous elments of S which are not in $\mathfrak{p}$. For any open subset $U \subset \mathrm{Proj}(S)$ we define $\mathcal{O}(U)$ to be the set of functions $s : U \to \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$ we have $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ and there is an open neighbourhood V of $\mathfrak{p}$ in U and homogeneous elements a, f in S of the

same degree such that for all $\mathfrak{q} \in V$ we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$. Prove the following:

(a) For any $\mathfrak{p} \in \operatorname{Proj}(S)$, the stalk $\mathcal{O}_{\mathfrak{p}}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$.

(b) For any homogeneous $f \in S_+$ let U_f^+ be the complement of $V((f))$. These open sets cover $\operatorname{Proj}(S)$ and there is an isomorphism of locally ringed spaces

$$(U_f^+, \mathcal{O}|_{U_f^+}) \cong \operatorname{Spec}(S_{(f)})$$

where $S_{(f)}$ is the subring of elements of degree zero in the localized ring S_f.

In particular $\operatorname{Proj}(S)$ is a scheme.

Solution:

(a) As in the affine case, we define a homomorphism $\varphi : \mathcal{O}_{\mathfrak{p}} \to S_{(\mathfrak{p})}$ by sending any local section s in a neighbourhood of $\mathfrak{p}$ to its value $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$. This is surjective, since every element $a/f \in S_{(\mathfrak{p})}$ for two homogeneous elements $a, f \in S$ of the same degree such that $f \notin \mathfrak{p}$, define a well-defined section $U_f \to \coprod S_{(\mathfrak{p})}$, whose image at $\mathfrak{p}$ is a/f. The map φ is also injective, because for any element $s_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$ such that $\varphi(s_{\mathfrak{p}}) = 0$ there is an open neighbourhood U of $\mathfrak{p}$ and a section s over U such that $s(\mathfrak{p}) = s_{\mathfrak{p}}$ and s is given by $\mathfrak{q} \mapsto a/f$ for some homogeneous elements $a, f \in S$ of the same degree and $f \notin \mathfrak{q}$ for all $\mathfrak{q} \in U$. Since $\varphi(s_{\mathfrak{p}}) = 0$ we conclude that there is a homogeneous element $u \notin \mathfrak{p}$ such that $ua = 0$. Thus $s|_{U_u \cap U} = 0$, which implies that $s_{\mathfrak{p}} = 0$. Hence φ is an isomorphism.

(b) Since $\operatorname{Proj}(S)$ is the set of all homogeneous prime ideals which do not contain S_+, the sets U_f^+ for homogeneous $f \in S_+$ cover $\operatorname{Proj}(S)$. Consider the localisation map $S \to S_f$. We know that $S_{(f)}$ is a subring of S_f. For a homogeneous ideal $\mathfrak{a} \subset S$ define $\varphi(\mathfrak{a}) := (\mathfrak{a}S_f) \cap S_{(f)}$. In particular for $\mathfrak{p} \in U_f^+$ we have $\varphi(\mathfrak{p}) \in \operatorname{Spec}(S_{(f)})$. This respects inclusions and is bijective, and so φ defines a homeomorphism $\varphi : U_f^+ \to \operatorname{Spec}(S_{(f)})$. Note that for $\mathfrak{p} \in U_f^+$ we have $S_{(\mathfrak{p})} \cong (S_{(f)})_{(\varphi(\mathfrak{p}))}$ and so the sheaf homomorphism is an isomorphism. This proves that $(U_f^+, \mathcal{O}|_{U_f^+}) \cong \operatorname{Spec}(S_{(f)})$.

8. We define projective n-space over a ring A as $\mathbb{P}_A^n := \operatorname{Proj}(A[X_0, \dots, X_n])$ with the definition of the previous exercise. Prove that $\mathbb{P}_A^n$ can also be seen as $n+1$ copies of affine n-space $\operatorname{Spec}(A[Y_1, \dots, Y_n])$ glued together in the right way. Concretely, find an open cover of $n+1$ subsets of $\mathbb{P}_A^n$ which are all isomorphic to a copy of affine n-space $\operatorname{Spec}(A[Y_1, \dots, Y_n])$. What are the transition morphisms between those charts?

Solution: We consider the $n+1$ open subsets $U_{X_0}^+, \dots, U_{X_n}^+$. Denote the ring $B := A[X_0, \dots, X_n]$. By the previous exercise we know $U_{X_i}^+ \cong \operatorname{Spec}(B_{(X_i)})$ for all

$0 \leqslant i \leqslant n$. We prove that $A[Y_1, \dots, Y_n] \cong B_{(X_i)}$. Define the ring homomorphism $A[Y_1, \dots, Y_n] \to B_{(X_i)}$ by

$$Y_k \mapsto \begin{cases} \frac{X_{k-1}}{X_i}, & 0 < k \leqslant i \\ \frac{X_k}{X_i}, & i < k \leqslant n \end{cases} \qquad \text{for } 1 \leqslant k \leqslant n.$$

One easily sees that this homomorphism is both injective and surjective and hence an isomorphism. It is convenient to write the ring $B_{(X_i)}$ as $A[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}]$, but keep in mind, that this ring has only n generators, since $\frac{X_i}{X_i} = 1$. In this notation, the transition maps are easy, since they are given by leaving X_i invariant, i.e. the transition map from the chart $U^+_{X_j}$ to the chart $U^+_{X_i}$ on their intersection is given by the pullback of the canonical map $A[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}]_{\frac{X_j}{X_i}} \to A[\frac{X_0}{X_j}, \dots, \frac{X_n}{X_j}]_{\frac{X_i}{X_j}}$. In the other notation, it needs case distinction in general. For $n = 1$ the transition map $A[Y, Y^{-1}] \mapsto A[Z, Z^{-1}]$ is given by $Y \mapsto Z^{-1}$.

A. *Direct Limit.* Let I be a partially ordered directed set, i.e. I is partially ordered and for every pair $i, j \in I$ there exists $k \in I$ such that $i \geqslant k$ and $j \geqslant k$. Let $(A_i)_{i \in I}$ be a family of abelian groups indexed by I. Suppose that for each pair $i, j \in I$ such that $i \geqslant j$ there exists a homomorphism $\mu_{ij} : A_i \to A_j$ satisfying the following two properties:

- For all $i \in I$ we have $\mu_{ii} = \mathrm{id}_{A_i}$.
- For $i, j, k \in I$ with $i \geqslant j \geqslant k$ we have $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$.

We call this collection (A_i, μ_{ij}) a direct system of abelian groups over I. Let C be the direct sum of all A_i and denote by D the $\mathbb{Z}$-submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ for $x_i \in A_i$ and $i, j \in I$ with $i \geqslant j$. We define the *direct limit* of the direct system (A_i, μ_{ij}), denoted by $\varinjlim A_i$ as the quotient $\mathbb{Z}$-module C/D. For every $i \in I$ there is a homomorphism $\nu_i : A_i \to \varinjlim A_i$. Prove the following properties of $\varinjlim A_i$:

(a) Elements $x \in \varinjlim A_i$ are equivalence classes containing (i, x_i) with $x_i \in A_i$ such that $(i, x_i) \sim (j, x_j)$ if and only if there exists a $k \in I$ with $i \geqslant k$ and $j \geqslant k$ and such that $\mu_{ik}(x_i) = \mu_{ik}(x_i)$.

(b) If each A_i is a ring and the homomorphisms are ring homomorphisms, then the direct limit has a natural ring structure such that ν_i is a ring homomorphism for all $i \in I$.

(c) An element $[(i, x_i)] \in \varinjlim A_i$ is zero if and only if there exists a $j \in I$ with $i \geqslant j$ such that $\mu_{ij}(x_i) = 0$.

(d) If B is a group and there are homomorphisms $\varphi_i : A_i \to B$ for all $i \in I$ such that $\varphi_i = \varphi_j \circ \mu_{ij}$ for all $i, j \in I$ with $i \geqslant j$, then there is a homomorphism $\varphi : \varinjlim A_i \to B$ such that $\varphi \circ \nu_i = \varphi_i$ for all $i \in I$.

In view of the above we conclude that the stalk of a sheaf $\mathscr{F}$ at a point x in a topological space X is the same as the direct limit over the rings $\mathscr{F}(U)$ for U an open neighbourhood of x.

Solution:

(a) By definition, an element $x \in \varinjlim A_i$ is given by a finite sum $x = a_{i_1} + \cdots + a_{i_s}$ of elements $a_{i_k} \in A_{i_k}$ modulo the equivalence relations in D. Since we only have finitely many elements and the index set is a directed set, there is an index $\ell \in I$ such that $i_k \geqslant \ell$ for all k. Now x is equivalent to $x_\ell := \mu_{i_1\ell}(a_{i_1}) + \cdots + \mu_{i_s\ell}(a_{i_s}) \in A_\ell$. Furthermore, the equivalence relation is a different formulation of the equivalence relation given by D.

(b) Let $x, y \in \varinjlim A_i$. By (a) we can find $x_i \in A_i$ and $y_j \in A_j$ such that $x \sim x_i$ and $y \sim y_j$. Let $k \in I$ such that $i \geqslant k$ and $j \geqslant k$. We define $xy := \nu_k(\mu_{ik}(x_i)\mu_{jk}(y_j))$. This definition does not depend on a representative of x or y because the homomorphisms μ are compatible. One can check that all ring axioms are satisfied and thus $\varinjlim A_i$ has a natural ring structure. Moreover, the ν_i are ring homomorphisms by design.

(c) This follows directly by (a).

(d) We define the homomorphism φ as $\varphi([(i, x_i)]) := \varphi_i(x_i)$ using (a). This is well-defined because of the compatibility condition of the φ_i. It follows directly that it is a homomorphism and that we have $\varphi \circ \nu_i = \varphi_i$ for all $i \in I$.