D-MATH
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Algebraic Geometry

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Solutions Sheet 4

Schemes

1. Consider the affine plane $\mathbb{A}^2_{\mathbb{R}}$ over $\mathbb{R}$. Show that the non-closed points are all either

   (a) the point (0), whose closure is all of $\mathbb{A}^2_{\mathbb{R}}$, or

   (b) the point (f) corresponding to an irreducible polynomial $f \in \mathbb{R}[X, Y]$.

   The polynomials of (b) may or may not remain irreducible in $\mathbb{C}[X, Y]$, so that a non-closed point in $\mathbb{A}^2_{\mathbb{R}}$ corresponds either to a single non-closed point of $\mathbb{A}^2_{\mathbb{C}}$ or to two non-closed points in $\mathbb{A}^2_{\mathbb{C}}$. The closed points in the closure of such a non-closed point may be either of both types above, or only of the second. Give examples of all these possibilities.

   Solution: Let $(0) \neq \mathfrak{p} \in \mathbb{A}^2_{\mathbb{R}}$ be a non-closed point of $\mathbb{A}^2_{\mathbb{R}}$. We define $n_p := \min_{g \in \mathfrak{p}} \deg(g)$. Now let $f \in \mathfrak{p}$ such that $\deg(f) = n_p$. Then f is irreducible: suppose $g, h \in \mathbb{R}[X, Y]$ such that $gh = f$, then $h \in \mathfrak{p}$ or $g \in \mathfrak{p}$ since $\mathfrak{p}$ is a prime ideal. Without loss of generality we may assume $g \in \mathfrak{p}$. By definition $\det(g) \leqslant n_{\mathfrak{p}} = \deg f$, hence $\deg(h) = 0$, i.e. h is a constant. Now let $\mathfrak{p} \subsetneq \mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$, then

   $$(0) \subsetneq (f) \subset \mathfrak{p} \subsetneq \mathfrak{m}$$

   is a chain of prime ideals in $\mathbb{R}[X, Y]$. Then $\mathfrak{p} = (f)$, since $\dim(\mathbb{R}[X, Y]) = 2$. Concerning the second part of the exercise it is easy to check that the non-closed point (X) corresponds to the non closed point (X) in $\mathbb{A}^2_{\mathbb{C}}$ while the non-closed point $(X^2 + 1)$ corresponds to the two closed points $(X - i)$ and $(X + i)$ in $\mathbb{A}^2_{\mathbb{C}}$.

2. An inclusion of fields $K \hookrightarrow L$ induces a morphism $\mathbb{A}^n_L \to \mathbb{A}^n_K$. Find the images of the following points under the morphism $\mathbb{A}^2_{\overline{\mathbb{Q}}} \to \mathbb{A}^2_{\mathbb{Q}}$:

   (a) $(X - \sqrt{2}, Y - \sqrt{2})$

   (b) $(X - \sqrt{2}, Y - \sqrt{3})$

   (c) $(X - \zeta, Y - \zeta^{-1})$, for ζ a p-th root of unity for p prime

   (d) $(\sqrt{2}X - \sqrt{3}Y)$

   (e) $(\sqrt{2}X - \sqrt{3}Y - 1)$

   Solution: Let us denote

   $$\begin{array}{rcl} i^* : \mathbb{A}^2_{\overline{\mathbb{Q}}} & \to & \mathbb{A}^2_{\mathbb{Q}} \\ \mathfrak{p} & \mapsto & i^{-1}(\mathfrak{p}), \end{array}$$

   where $i : \mathbb{Q}[X, Y] \hookrightarrow \overline{\mathbb{Q}}[X, Y]$.

(a) By definition we have

$$\mathfrak{q} := i^{-1}(X-\sqrt{2}, Y-\sqrt{2}) = \{f \in \mathbb{Q}[X,Y] : X-\sqrt{2}|f \text{ or } Y-\sqrt{2}|f \text{ in } \overline{\mathbb{Q}}[X,Y]\}.$$

Then $X^2-2 = (X-\sqrt{2})(X+\sqrt{2}) \in \mathfrak{q}$ and $Y^2-2 = (Y-\sqrt{2})(Y+\sqrt{2}) \in \mathfrak{q}$. So we have $(X^2-2, Y^2-2) \subset \mathfrak{q}$. On the other hand we have

$$\mathbb{Q}[X,Y]/(X^2-2, Y^2-2) = \mathbb{Q}[\sqrt{2}],$$

i.e. (X^2-2, Y^2-2) is a maximal ideal. Hence $\mathfrak{q} = (X^2-2, Y^2-2)$.

(b) with as similar argument as in (a) one shows that $i^{-1}(X-\sqrt{2}, Y-\sqrt{3}) = (X^2-2, Y^2-3)$.

(c) with as similar argument as in (a) one shows that $i^{-1}(X-\zeta, Y-\zeta^{-1}) = (\Phi_p(X), \Phi_p(Y))$, Where $\Phi(Z) = \sum_{0\leqslant n<p} Z^n$ is the p-th Cyclotomic polynomial

(d) By definition we have

$$\mathfrak{q} := i^{-1}(\sqrt{2}X-\sqrt{3}Y) = \{f \in \mathbb{Q}[X,Y] : (\sqrt{2}X-\sqrt{3}Y)|f \text{ in } \overline{\mathbb{Q}}[X,Y]\}.$$

Then $2X^2-3Y^2 = (\sqrt{2}X-\sqrt{3}Y)(\sqrt{2}X+\sqrt{3}Y) \in \mathfrak{q}$. On the other hand $2X^2-3Y^2$ is irreducible over $\mathbb{Q}[X,Y]$, thus $(2X^2-3Y^2)$ is a prime ideal in $\mathbb{Q}[X,Y]$. We consider the chain of prime ideal

$$(0) \subsetneq (2X^2-3Y^2) \subset \mathfrak{q} \subsetneq (X,Y),$$

where the last strict inclusion it is true since $(\sqrt{2}X-\sqrt{3}Y) \subsetneq (X,Y)$ in $\overline{\mathbb{Q}}[X,Y]$ and $X, Y \notin \mathfrak{q}$. Using the fact that $\dim(\mathbb{Q}[X,Y]) = 2$ we have that $\mathfrak{q} = (2X^2-3Y^2)$.

(e) Using the fact that

$$\begin{aligned}4X^4-4X^2+9Y^2-6Y-12X^2Y+1 &= (\sqrt{2}X-\sqrt{3}Y-1)(\sqrt{2}X+\sqrt{3}Y-1)\\ &\cdot(-\sqrt{2}X-\sqrt{3}Y-1)(-\sqrt{2}X+\sqrt{3}Y-1),\end{aligned}$$

and the same argument as in (d) one shows that $i^{-1}(\sqrt{2}X-\sqrt{3}Y-1) = (4X^4-4X^2+9Y^2-6Y-12X^2Y+1)$.

3. *Double Points.* Let k be a field and $Y \hookrightarrow \mathbb{A}^2_k$ be a closed subscheme with image containing the origin $(0,0)$ in $\mathbb{A}^2_k$ and such that $\mathcal{O}_Y(Y) \cong k[\varepsilon]/(\varepsilon^2)$. Denote by $\varphi : k[x,y] \to \mathcal{O}_Y(Y)$ the surjection defining the inclusion $Y \hookrightarrow \mathbb{A}^2$. Prove that the kernel of φ contains a non-zero element $\alpha x + \beta y$ for some $\alpha, \beta \in k$. Write $X_{\alpha,\beta} := \operatorname{Spec}(k[x,y]/\ker(\varphi))$ and show that $X_{\alpha,\beta}$ can also be characterized as the composition of the natural morphism $\operatorname{Spec}(k[\varepsilon]/(\varepsilon^2)) \to \operatorname{Spec}(k[\varepsilon]) \cong \mathbb{A}^1_k$ with the inclusion of the line $\mathbb{A}^1_k \hookrightarrow \mathbb{A}^2_k$ given by $x \mapsto (\beta x, -\alpha x)$.

Solution: You find this solution in the Book of Eisenbud and Harris, Examples $II.9$, $II.10$.

4. Let k be an algebraically closed field and let $Z := \mathrm{Spec}(k[X_1, \ldots, X_n]/I) \subset \mathbb{A}^n_k$ be a closed subscheme of dimension 0 supported at the origin (i.e. $\sqrt{I} = (X_1, ..., X_n)$). Furthermore, suppose that $k[X_1, \ldots, X_n]/I$ is a 3-dimensional k-vector space. Prove that Z is isomorphic to either $A := \mathrm{Spec}(k[X]/(X^3))$ or to $B := \mathrm{Spec}(k[X, Y]/(X^2, XY, Y^2))$ and that A and B are not isomorphic to each other.

Solution: Let us start proving that A and B are not isomorphic. To show this it is enough to prove that $k[X]/(X^3)$ and $k[X, Y]/(X^2, XY, Y^2)$ are not isomorphic as rings. Let $\varphi : k[X]/(X^3) \to k[X, Y]/(X^2, XY, Y^2)$ be any ring homomorphism from $k[X]/(X^3)$ to $k[X, Y]/(X^2, XY, Y^2)$. Then $\varphi(X^2) = \varphi(X)^2 = 0$. On the other hand $X^2 \neq 0$ in A, thus $\mathrm{Ker}(\varphi) \neq 0$. It follows that $k[X]/(X^3)$ and $k[X, Y]/(X^2, XY, Y^2)$ can not be isomorphic since any homorphism from from $k[X]/(X^3)$ to $k[X, Y]/(X^2, XY, Y^2)$ is not injective.

Let us prove the other part of the Exercise. in the following, x_i denotes the imagine of X_i in $k[X_1, \ldots, X_n]/I$. Since $\sqrt{I} = (X_1, ..., X_n)$, then $\mathrm{Spec}(k[X_1, \ldots, X_n]/I) = \{\mathfrak{m}\}$ where $\mathfrak{m}$ is the reduction of $(X_1, ..., X_n)$ modulo I. Moreover we have that

$$k[X_1, \ldots, X_n]/I = k \oplus \mathfrak{m}$$

thus $\dim_k(\mathfrak{m}) = 2$. We claim that $\mathfrak{m}^3 = 0$. Let us consider the chain

$$\mathfrak{m} \supset \mathfrak{m}^2 \supset \mathfrak{m}^3.$$

Then $\mathfrak{m} \supsetneq \mathfrak{m}^2$, otherwise by Nakayama's Lemma $\mathfrak{m} = 0$. Thus $\dim_k(\mathfrak{m}^2) \leqslant 1$ If $\dim_k(\mathfrak{m}^2) = 0$ we have done, otherwise $\mathfrak{m}^2 \supsetneq \mathfrak{m}^3$ (again thanks to Nakayama'sd Lemma) and then $\mathfrak{m}^3 = 0$. Now we have to distinguish to situation

i) $\mathfrak{m}^2 = 0$. In this case we have that $\mathfrak{m} = \mathrm{span}_k\{x_1, ..., x_n\}$. Because $\dim_k(\mathfrak{m}) = 2$, without loss generalities we may assume that $\mathrm{span}\{x_1, x_2\}$. On the other hand $\mathfrak{m}^2 = 0$ implies that $x_1^2 = x_1x_2 = x_2^2 = 0$. Hence the map

$$\begin{array}{ccc} k[X, Y]/(X^2, Y^2, XY) & \to & k[X_1, ..., X_n]/I \\ X & \mapsto & x_1 \\ Y & \mapsto & x_2, \end{array}$$

is a ring isomorphism and so $Z \cong B$.

ii) $\mathfrak{m}^2 \neq 0$. We claim that there exists $i \in \{1, ..., n\}$ such that $x_i^2 \neq 0$. By contradiction assume this is not the case. Let x_i, x_j with $i \neq j$ such that $x_ix_j \neq 0$ (we know that there exists at least one of these pairs because $\mathfrak{m}^2 \neq 0$ and we are assuming $x_i^2 = 0$ for any $i = 1, ..., n$). Then $\mathrm{span}_k\{x_i, x_ix_j\} = \mathfrak{m}$: let $\alpha, \beta \in k$ such that

$$\alpha x_i + \beta x_i x_j = 0,$$

then

$$x_i(\alpha + \beta x_i x_j) = 0.$$

This implies $\alpha = 0$, otherwise $\alpha + \beta x_i x_j$ is invertible in $k[X_1, \ldots, X_n]/I$ and so $x_i = 0$ and this is not possible since we are assuming $x_i x_j = 0$. On the other hand $\alpha = 0$ implies $\beta = 0$ again because $x_i x_j \neq 0$. Then $\dim(\mathrm{span}_k\{x_i, x_i x_j\}) = 2$ and so $\mathrm{span}_k\{x_i, x_i x_j\} = \mathfrak{m}$. In particular there exist $\alpha, \beta \in k$ such that

$$x_j = \alpha x_i + \beta x_i x_j.$$

Multiplying both sides by x_j we get

$$x_j^2 = \alpha x_i x_j + \beta x_i x_j^2,$$

thus $\alpha x_i x_j = 0$ which implies $\alpha = 0$. So we have

$$x_j = \beta x_i x_j.$$

implies $x_j(1-\beta x_i) = 0$. On the other hand $1-\beta x_i$ is invertible in $k[X_1, \ldots, X_n]/I$, thus $x_j = 0$ and then $x_i x_j = 0$ and this is absurd since we are assuming $x_i x_j \neq 0$. Hence, without loss of generality we may assume that x_1 is such that $x_1^2 \neq 0$. Then $\mathrm{span}_k\{x_1, x_1^2\} = \mathfrak{m}$: indeed if $x_1^2 = \lambda x_1$ for some $\lambda \in k^\times$ then $x_1^3 = \lambda^2 x_1 \neq 0$ since $x_1 \neq 0$. On the other hand $x_1^3 \in \mathfrak{m}^3 = 0$ and this leads to a contradiction. Thus $x_1^2 \neq 0$ and $x_1^2 \notin \mathrm{span}_k\{x_1\}$, i.e. $\mathrm{span}_k\{x_1, x_1^2\} = \mathfrak{m}$. Hence the map

$$\begin{array}{ccc} k[X]/(X^3) & \to & k[X_1, ..., X_n]/I \\ X & \mapsto & x_1 \end{array}$$

is a ring isomorphism and so $Z \cong A$.

Let us consider $V = \mathrm{span}\{x_1, ..., x_n\}$, we have two possibilities:

i) $\dim_k(V) = 2$. With out loss of generalities we may assume $V = \mathrm{span}\{x_1, x_2\}$. Then $k[X_1, \ldots, X_n]/I = \mathrm{span}\{1, x_1, x_2\}$

ii) $\dim_k(V) = 1$

5. Let $X := \mathbb{A}^2_{\mathbb{C}} \setminus \{0\} \subset \mathbb{A}^2_{\mathbb{C}}$. Prove:

   (a) The restriction map $\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}(\mathbb{A}^2_{\mathbb{C}}) \to \mathcal{O}_X(X)$ is an isomorphism.

   (b) The scheme X is not an affine scheme.

Solution:

(a) We compute $\Gamma(\mathbb{A}^2_{\mathbb{C}} \setminus \{0\}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}})$. Since

$$\mathbb{A}^2_{\mathbb{C}} \setminus \{0\} = \mathbb{A}^2_{\mathbb{C},X} \cup \mathbb{A}^2_{\mathbb{C},Y}$$

where

$$\mathbb{A}^2_{\mathbb{C},X} := \mathbb{A}^2_{\mathbb{C}} \setminus \{\mathfrak{p} : (X) \subset \mathfrak{p}\}, \quad \mathbb{A}^2_{\mathbb{C},Y} := \mathbb{A}^2_{\mathbb{C}} \setminus \{\mathfrak{p} : (Y) \subset \mathfrak{p}\},$$

we have that $\Gamma(\mathbb{A}^2_{\mathbb{C}} \setminus \{0\}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}})$ is the kernel of the map

$$\begin{array}{rcl} \Gamma(\mathbb{A}^2_{\mathbb{C},X}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) \oplus \Gamma(\mathbb{A}^2_{\mathbb{C},Y}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) & \to & \Gamma(\mathbb{A}^2_{\mathbb{C},XY}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) \\ (s,t) & \mapsto & s_{|\mathbb{A}^2_{\mathbb{C},XY}} - t_{|\mathbb{A}^2_{\mathbb{C},XY}}, \end{array}$$

where $\mathbb{A}^2_{\mathbb{C},XY} = \mathbb{A}^2_{\mathbb{C},X} \cap \mathbb{A}^2_{\mathbb{C},Y}$. Let $s \in \Gamma(\mathbb{A}^2_{\mathbb{C},X}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}})$ and $t \in \Gamma(\mathbb{A}^2_{\mathbb{C},Y}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}})$, since $\Gamma(\mathbb{A}^2_{\mathbb{C},X}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) = k[X,Y]_X$, $\Gamma(\mathbb{A}^2_{\mathbb{C},Y}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) = k[X,Y]_Y$ there exist two polynomial $f, g \in k[X,Y]$ such that $X \nmid f$, $Y \nmid g$ and $s = \frac{f}{X^m}$, $g = \frac{t}{Y^n}$ for some $m, n \geqslant 0$. On the other hand $\Gamma(\mathbb{A}^2_{\mathbb{C},XY}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) = k[X,Y]_{XY}$, thus $s_{|\mathbb{A}^2_{\mathbb{C},XY}} - t_{|\mathbb{A}^2_{\mathbb{C},XY}} = 0$ if and only if

$$f \cdot Y^n = g \cdot X^m,$$

and this is possible if and only if $n = m = 0$ and $f = g$. Hence $\Gamma(\mathbb{A}^2_{\mathbb{C}} \setminus \{0\}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}) = k[X,Y]$.

(b) By contradiction, assume $\mathbb{A}^2_{\mathbb{C}} \setminus \{0\}$ affine scheme, then

$$\mathbb{A}^2_{\mathbb{C}} \setminus \{0\} = \operatorname{Spec}(\Gamma(\mathbb{A}^2_{\mathbb{C}} \setminus \{0\}, \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}})) = \operatorname{Spec}(k[X,Y]) = \mathbb{A}^2_{\mathbb{C}},$$

and this is absurd.

6. Let X be a scheme and $f \in \mathcal{O}_X(X)$ a global section. Define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$.

   (a) If $U = \operatorname{Spec}(B)$ is an open affine subscheme of X and if $\bar{f} \in B = \mathcal{O}_U(U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.

   (b) Assume that X is quasi-compact. Let $A := \mathcal{O}_X(X)$ and let $a \in A$ be an element whose restriction to X_f is 0. Show that there exists an integer $n > 0$ such that $f^n a = 0$.

   (c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. Let $b \in \mathcal{O}_{X_f}(X_f)$. Show that there exists an integer $n > 0$ such that $f^n b$ is the restriction of an element of A.

   (d) With the hypothesis of (c) conclude that $\mathcal{O}_{X_f}(X_f) \cong A_f$.

Solution:

(a) We have that

$$\begin{aligned} X_f \cap U := & \{\mathfrak{p} \in \operatorname{Spec}(B) : f_{\mathfrak{p}} \notin \mathfrak{p}B_{\mathfrak{p}}\} \\ = & \{\mathfrak{p} \in \operatorname{Spec}(B) : \overline{f} \notin \mathfrak{p}B_{\mathfrak{p}}\} \\ = & \operatorname{Spec}(B_{\overline{f}}), \end{aligned}$$

thus $X_f \cap U = D(\overline{f})$ has we wanted.

(b) Since X is quasi-compact by hypothesis, we can find a finite affine cover, $\{U_i\}_{i=1}^n$, of X. For any $i = 1, ..., n$, let B_i denote the ring such that $U_i = \operatorname{Spec}(B_i)$. Let $a \in \mathcal{O}_X(X)$ such that $a_{|X_f} = 0$. For any $i = 1, ..., n$ we have that $a_{|X_f \cap U_i} = 0$. On the other hand by part (a) we know that $X_f \cap U_i = (U_i)_{f_i} = \operatorname{Spec}(B^i_{f_i})$, where f_i denote the restriction of f to U_i. Thus for $i = 1, ..., n$, $a_{|(U_i)_{f_i}} = 0$ and this implies that there exist $n_i \geqslant 0$ such that

$$f_i^{n_i} \cdot a_{|U_i} = 0 \text{ in } B_i.$$

Let us take $N := \max_{1 \leqslant i \leqslant n} n_i$. For any $i = 1, ..., n$ one has that $f^N a_{|U_i} = f_i^N \cdot a_{|U_i} = 0$. Since $\{U_i\}_{i=1}^n$ is a cover of X and $\mathcal{O}_X$ is a sheaf we conclude that $f^N a = 0$.

(c) Let $b \in \mathcal{O}_{X_f}(X_f)$, $\{U_i\}_{i=1}^n$ be a finite affine cover such that for any i, j, $U_i \cap U_j$ is quasi-compact and for any $i = 1, ..., n$, let b_i denotes the restriction of b to $X_f \cap U_i = (U_i)_{f_i}$. Since $(U_i)_{f_i} = \operatorname{Spec}((B_i)_{f_i})$, there exists $n_i \geqslant 0$ such that $f_i^{n_i} b_i \in B_i$. Let us take $N := \max_{1 \leqslant i \leqslant n} n_i$. Fix i, j and consider

$$b_{i,j} := (f_i^N b_i)_{|U_i \cap U_j} - (f_j^N b_j)_{|U_i \cap U_j} \in \mathcal{O}_X(U_i \cap U_j).$$

It is easy to see that $b_{ij||U_i \cap U_j \cap X_f} = 0$, then we can apply part (b) since $U_i \cap U_j$ is quasi compact. Thus there exists $n_{i,j} \geqslant 0$ such that

$$f^{n_{i,j}}_{|U_i \cap U_j} c_{i,j} = 0.$$

Let us take $M := \max_{i,j} n_{i,j}$, We claim that $f^{N+M} b \in \mathcal{O}_X(X)$. For any i, $f_i^{N+M} b_i \in \mathcal{O}_X(U_i)$, such that $(f_i^{N+M} b_i)_{|U_i \cap U_j} = (f_j^{N+M} b_j)_{|U_i \cap U_j}$ for any i, j. Thus $f^{N+M} b \in \mathcal{O}_X(X)$ since $\mathcal{O}_X$ is a sheaf.

(d) The inclusion $A_f \subset \mathcal{O}_{X_f}(X_f)$ follows because for any $i = 1, ..., n$, $\{X_f \cap U_i\}_{i=1}^n$ is a covering of open affine subset of X_f and $\mathcal{O}_{X_f}(X_f \cap U_i) = B^i_f$. For the other inclusion, let $s \in \mathcal{O}_{X_f}(X_f)$ then by part (c) there exists $n \geqslant 0$ such that $a := f^n s \in \mathcal{O}_X(X)$. On the other hand f is invertible in $\mathcal{O}_{X_f}(X_f)$, thus $s = \frac{a}{f^n}$ as we wanted.

7. *A Criterion for Affineness.*

(a) Let $f : X \to Y$ be a morphism of schemes and suppose that Y can be covered by open subsets U_i such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

(b) A scheme X is affine if and only if there is a finite set of elements $f_1, \dots, f_r \in A := \mathcal{O}_X(X)$ such that the open subsets X_{f_i} defined in exercise 6 are affine and $f_1, \dots, f_r$ generate the unit ideal.

Solution:

(a) Let $f : X \to Y$ a map as in the statement of the exercise. To check that f is an isomorphis it is enough to check that for any $x \in X$ the map

$$f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is an isomorphism. Let U_i be an open set in the covering such that $x \in U_i$, then we have that

$$(f_{|U_i})_x : (\mathcal{O}_{Y|_{f(U_i)}})_{f(x)} \to (\mathcal{O}_{X|_{U_i}})_x$$

is an isomorphism by hypothesis. On the other hand we have $(\mathcal{O}_{Y|_{f(U_i)}})_{f(x)} \cong \mathcal{O}_{Y,f(x)}$ and $(\mathcal{O}_{X|_{U_i}})_x \cong \mathcal{O}_{X,x}$ since U_i and $f(U_i)$ are open subset respectively of X and Y.

(b) First We prove that $\cup_{i=1}^n X_{f_i}$. By contradiction, assume that there exists $x \in X \setminus \cup_{i=1}^n X_{f_i}$, then $f_i \in \mathfrak{m}_x$ for any $i = 1, ..., n$. This would implies that $1 \in \mathfrak{m}_x$ and this is not possible since $\mathcal{O}_{X,x}$ has to be a local ring. Because we are assuming X_{f_i} to be affine we can apply part (d) of the previous exercise getting $X_{f_i} = \mathrm{Spec}(A_{f_i})$ for any $i = 1, ..., n$. From the ring homomorphisms

$$\begin{array}{rcl} g_i^* : A & \to & A_{f_i} \\ a & \mapsto & \frac{a}{1}, \end{array}$$

we get the scheme morphisms

$$g_i : X_{f_i} \to \mathrm{Spec}(A)$$

which are isomorphisms between X_f and $g_i(X_{f_i}) = \mathrm{Spec}(A_{f_i})$. Gluing together these maps we obtain a morphism

$$g : X \to \mathrm{Spec}(A).$$

Then we conclude applying part (a).

8. Let k be a field. We want to study $\mathrm{Spec}\, A$ where $A = k(u) \otimes_k k(v)$ is the tensor product of two purely trascendental extension of k of trascendence degree 1 (i.e. $k(u) \cong k(X)$ the field of rational function at coefficient in k).

(a) We have $k(u) = T^{-1}k[u]$ where T is the multiplicative set made up of the non-zero elements of $k[u]$. Deduce from this that A is the localization of $k[u, v]$ with respect to the multiplicative set T' made up of the non-zero elements of the form $P(u)Q(v) \in k[u, v]$.

(b) Let $\mathfrak{m}$ be a maximal ideal of $k[u,v]$. Show that there exist a $P(u) \in \mathfrak{m} \setminus \{0\}$. Deduce from this that $T' \cap \mathfrak{m} \neq \varnothing$.

(c) Show that the maximal ideals of A are of the form gA with $g \in k[u,v] \setminus (k[u] \cup k[v])$ irreducible in $k[u,v]$.

(d) Show that $\operatorname{Spec} A$ is an infinite set and that $\dim A = 1$.

Solution:

(a) By definition

$$T^{-1}k[u] := \left\{ \frac{f}{g} : f, g \in k[u],\ g \neq 0 \right\} = \operatorname{Frac}(k[u]) = k(u).$$

We denote by T_u (respectively by T_v) the multiplicative set made up of the non-zero elements of $k[u]$ (respectively the multiplicative set made up of the non-zero elements of $k[v]$). It is a well known fact that the map

$$\begin{array}{rcl} \varphi : k[u] \otimes k[v] & \to & k[u,v] \\ f(u) \otimes g(v) & \mapsto & f(u) \cdot g(v) \end{array}$$

is a isomorphism. Since localization is an exact functor, $k(u) \otimes_{k[v]}$ is flat and we get that

$$\begin{array}{rcl} 1_{k(u)} \otimes \varphi : k(u) \otimes k[v] & \to & T_u^{-1}k[u,v] \\ f(u) \otimes g(v) & \mapsto & f(u) \cdot g(v), \end{array}$$

is still an isomorphism. On the other hand, the same holds for $\otimes_{k[v]} k(v)$ so

$$\begin{array}{rcl} 1_{k(u)} \otimes \varphi \otimes 1_{k(v)} : k(u) \otimes k(v) & \to & T_v^{-1}T_u^{-1}k[u,v] \\ f(u) \otimes g(v) & \mapsto & f(u) \cdot g(v), \end{array}$$

is an isomorphism. Then we conclude thanks to the fact that $T_u \cdot T_v = T'$.

(b) Let $\mathfrak{m}$ be a maximal ideal of $k[u,v]$. Then $k[u,v]/\mathfrak{m}$ is a noetherian algebra over k of dimension 0 (since $\operatorname{ht}(\mathfrak{m}) = 2$). i.e, $k[u,v]/\mathfrak{m}$ is an artinian algebra over k (see Atiyah-Macdonald Proposition 8.5). This is equivalent to be a finite k-algebra (see Atiyah-Macdonald page 92 Exercise 3), i.e. $k[u,v]/\mathfrak{m}$ k-vector space of dimension m for some $m < \infty$. Let $\overline{u}$ be the image of u in $k[u,v]/\mathfrak{m}$. Then $\{1, \overline{u}, ..., \overline{u}^m\}$ is a set $m+1$ vectors in a m dimensional k-vector space, then there exist $a_0, ..., a_{m-1} \in k$ such that

$$\overline{u}^m = \sum_{i=0}^{m-1} a_i \overline{u}^i.$$

Thus we conclude that

$$P(u) := u^m - \sum_{i=0}^{m-1} a_i u^i \in \mathfrak{m},$$

as we wanted.

(c) By part (a) we know that $A \cong T'^{-1}k[u,v]$ where T' is the multiplicative set made up of the non-zero elements of the form $P(u)Q(v) \in k[u,v]$, thus

$$\operatorname{Spec} A = \{\mathfrak{p} \in \operatorname{Spec}(k[u,v]) : T' \cap \mathfrak{p} = \varnothing\}.$$

On the other hand an application of the Krull's principal ideal Theorem implies that

$$\operatorname{Spec}(k[u,v]) = \{(0)\} \cup \{(f) : f \in k[u,v] \text{ irreducible}\} \cup \{\mathfrak{m} \in \operatorname{Spec}(k[u,v]) : \mathfrak{m} \text{ maximal}\}.$$

Then using part (b) the result follows.

(d) First we show that $\dim A = 1$. We denote by

$$\begin{array}{rcl} i : k[u,v] & \to & T'^{-1}k[u,v] \\ f & \mapsto & \frac{f}{1}, \end{array}$$

The inclusion of $k[u,v]$ in $T'^{-1}k[u,v]$. Let $\mathfrak{p} \in \operatorname{Spec} A$ be a non zero ideal. Then $i^{-1}(\mathfrak{p}) \in \operatorname{Spec}(k[u,v])$ contains an irreducible polynomial g (see Exercise 1). On the other hand $g \notin T'$ since $i^{-1}(\mathfrak{p}) \cap T' = \varnothing$. Hence $\mathfrak{p} \supset gA$ and this implies that $\mathfrak{p} = gA$ is a maximal ideal thanks to part (c). We have shown that any non zero prime ideal in A is maximal, i.e. $\dim A = 1$. To show that $\operatorname{Spec} A$ is an infinite set it is enough to consider the family of ideals $((u+v^n)A)_{n\in\mathbb{N}}$: it is easy to show that for any $n \geqslant 1$ $u+v^n \notin T'$ and $u+v^n$ irreducible.

9. In the following if X is a scheme we denote by $\operatorname{sp}(X)$ the underlying topological space of X. Let S be a scheme and $\pi : X \to S$, $\rho : Y \to S$ be S-schemes. Let $\operatorname{sp}(X) \times_{\operatorname{sp}(S)} \operatorname{sp}(Y)$ be the fiber product of sets defined by π and ρ, endowed with the topology induced by the product topology on $\operatorname{sp}(X) \times \operatorname{sp}(Y)$. We are going to study some property concerning the relation between $\operatorname{sp}(X \times Y)$ and $\operatorname{sp}(X) \times_{\operatorname{sp}(S)} \operatorname{sp}(Y)$.

   (a) Show that we have a canonical map $f : \operatorname{sp}(X \times_S Y) \to \operatorname{sp}(X) \times_{\operatorname{sp}(S)} \operatorname{sp}(Y)$.

   (b) Show that f is surjective.

   (c) Let us consider the example $X = Y = \operatorname{Spec}\mathbb{C}$ and $S = \operatorname{Spec}\mathbb{R}$. Show that $X \times_S Y \cong \operatorname{Spec}(\mathbb{C} \oplus \mathbb{C})$ and that f is not injective.

   (d) Show that in the case of the previous Exercise, with $X = \operatorname{Spec} k(u)$, $Y = \operatorname{Spec} k(v)$ and $S = \operatorname{Spec} k$, the map f has infinite fibers.

   (e) Let $S = \operatorname{Spec} k$ be the spectrum of an arbitrary field. By studying the example $X = Y = \mathbb{A}^1_k$, show that the image of an open subset under f is not necessarily an open subset.

Solution:

(a) By definition of the fiber product of scheme we have two morphism $\lambda_1 : X \times_S Y \to X$ and $\lambda_2 : X \times_S Y \to Y$ such that diagram

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\lambda_1} & X \\ \downarrow{\scriptstyle \lambda_2} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\rho} & S \end{array}$$

is commutative. Then we define

$$\begin{array}{rcl} f : \mathrm{sp}(X \times Y) & \to & \mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y) \\ z & \mapsto & (\lambda_1(z), \lambda_2(z)). \end{array}$$

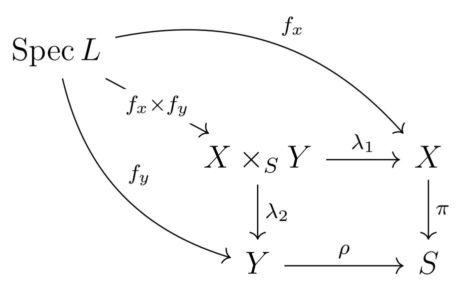
(b) Let $(x, y) \in \mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$ we want to show that there exist $z \in \mathrm{sp}(X \times_S Y)$ such that $f(z) = (x, y)$. Let us denote by $k(x), k(y)$ the residue field of x, y respectively. Since $(x, y) \in \mathrm{sp}(X) \times_{\mathrm{sp}(S)} \mathrm{sp}(Y)$ we have that $\pi(x) = \rho(y) = s \in S$. Moreover we also get $k(s) \hookrightarrow k(x)$ and $k(s) \hookrightarrow k(y)$ since $\pi : X \to S$ and $\rho : Y \to S$ are morphisms. Since $k(x), k(y) \supset k(s)$ we get that $k(x) \otimes_{k(s)} k(y) = k(x).k(y)$ the compositum field of $k(x)$ and $k(y)$. We denote $L := k(x)k(y)$ the compositum of $k(x)$ and $k(y)$. So we get the two morphisms

$$f_x : \mathrm{Spec}\, L \to X \qquad f_y : \mathrm{Spec}\, L \to Y,$$

such that $f_x((0)) = x$ and $f_y((0)) = y$. Thus we have the commutative diagram of scheme

$$\begin{array}{ccc} \mathrm{Spec}\, L & \xrightarrow{f_x} & X \\ \downarrow{\scriptstyle f_y} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\rho} & S. \end{array}$$

Using the the universal property of $X \times_S Y$ there exists a morphism of scheme $f_x \times f_y : \mathrm{Spec}\, L \to X \times_S Y$ such that



is a commutative diagram. Then $z := f_x \times f_y((0))$ is such that $f(z) = (x, y)$, indeed

$$\lambda_1(z) = \lambda_1 \circ f_x \times f_y((0)) = f_x((0)) = x$$

and similarly for λ_2 and y.

(c) Let $X = Y = \operatorname{Spec}\mathbb{C}$ and $S = \operatorname{Spec}\mathbb{R}$. To show that $X \times_S Y \cong \operatorname{Spec}(\mathbb{C} \oplus \mathbb{C})$ it is enough to show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$. An explicit isomorphism is given for example by

$$\begin{array}{rcl} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} & \to & \mathbb{C} \oplus \mathbb{C} \\ z \otimes w & \mapsto & (z \cdot w, z \cdot \overline{w}). \end{array}$$

Then f is not injective since $|\mathrm{sp}(\operatorname{Spec}\mathbb{C}) \times_{\mathrm{sp}(\operatorname{Spec}\mathbb{R})} \mathrm{sp}(\operatorname{Spec}\mathbb{C})| = 1$ while $|\mathrm{sp}(\operatorname{Spec}\mathbb{C} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C})| = 2$.

(d) We have that $\mathrm{sp}(\operatorname{Spec}(k(u))) \times_{\mathrm{sp}(\operatorname{Spec} k)} \mathrm{sp}(\operatorname{Spec}(k(v))) = \{((0),(0)\}$, thus $f^{-1}((0),(0)) = \operatorname{Spec} A$. On the other hand $|\operatorname{Spec} A| = \infty$ thanks to part (d) of the previous exercise.

(e) Since f is surjective it is enough to show that the image of a closed subset under f is not necessarily closed. Let $X = Y = \mathbb{A}^1_k$, then we know that $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathbb{A}^2_k$ with respect to the Zarisky topoly. On the other hand the topology in $\mathrm{sp}(\mathbb{A}^1_k) \times_{\mathrm{sp}(\operatorname{Spec} k)} \mathrm{sp}(\mathbb{A}^1_k)$ is the product topology, i.e. $Z \subset \mathrm{sp}(\mathbb{A}^1_k) \times_{\mathrm{sp}(\operatorname{Spec} k)} \mathrm{sp}(\mathbb{A}^1_k)$ is closed if and only if $Z = V(I) \times_{\mathrm{sp}(\operatorname{Spec} k)} V(J)$, where $I, J \subset k[X]$ are ideals. Consider $g = X + Y$. Then we have that

$$f(V((g))) = \{((X+a)(X-a)) : a \in k\} \cup \{((0)(0))\}$$

which is not of the for $V(I) \times_{\mathrm{sp}(\operatorname{Spec} k)} V(J)$. Thus $g(V((g))$ is not closed.