D-MATH Algebraic Geometry FS 2018

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Solutions Sheet 5

Projective Space, Divisors and the Picard group

1. Let $n \geqslant 1$ be an integer. Let A be a commutative ring. Prove that there is a canonical isomorphism $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) \to A$. [Hint: view the projective space as obtained by gluing $\mathrm{Spec}(A[x_0/x_i, \dots, x_n/x_i])$ for $0 \leqslant i \leqslant n$.]

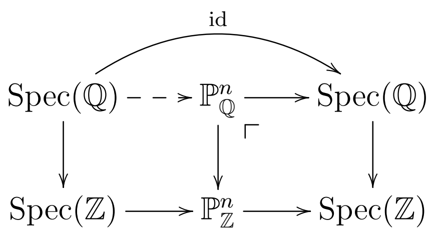
   Solution: By the gluing construction, a global section $f \in \Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A})$ is given by $n+1$ sections $f_i \in \Gamma(D(x_i), \mathcal{O}_{\mathbb{P}^n_A}) = A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$ such that $f_i|_{D(x_i)\cap D(x_j)} = f_j|_{D(x_i)\cap D(x_j)}$. Now fix $i \neq j$. We know that restriction to $D(x_i) \cap D(x_j)$ is the localisation map and by the gluing condition we get $f_i(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}) = f_j(\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j})$ which implies that $f_i = f_j \in A$. Doing this for all $i \neq j$ we conclude that $f_0 = \cdots = f_n \in A$. On the other hand, every sequence of sections $f_0 = \cdots = f_n \in A$ is compatible with the gluing, such that it lifts to a global section of $\mathbb{P}^n_A$.

2. (a) Construct a natural bijection $\mathbb{P}^n_{\mathbb{Z}}(\mathbb{Z}) \to \mathbb{P}^n_{\mathbb{Q}}(\mathbb{Q})$.

   (b) For a prime number p, compute the number of elements of $\mathbb{P}^n_{\mathbb{F}_p}(\mathbb{F}_p)$.

   Solution:

   (a) We know that $\mathbb{P}^n_{\mathbb{Q}} = \mathbb{P}^n_{\mathbb{Z}} \times \mathrm{Spec}(\mathbb{Q})$. Hence by the universal property of the fiber product, every $\mathbb{Z}$-point of $\mathbb{P}^n_{\mathbb{Z}}$ gives a unique $\mathbb{Q}$-point of $\mathbb{P}^n_{\mathbb{Q}}$ as the following diagram shows:



   So we only need to show that every $\mathbb{Q}$-point comes from this construction. Let $\mathrm{Spec}(\mathbb{Q}) \to \mathbb{P}^n_{\mathbb{Q}}$ be such a $\mathbb{Q}$-point. Since $\mathrm{Spec}(\mathbb{Q})$ is only one point its image lies in a chart $U_{x_i,\mathbb{Q}} = \mathrm{Spec}(\mathbb{Q}[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}])$ and thus induces a ring homomorphism $\mathbb{Q}[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \to \mathbb{Q}$ which is given by the image of $x_0, \dots, x_n$ in $\mathbb{Q}$ which we denote by $a_0, \dots, a_n$. Because we only consider ratios $\frac{x_j}{x_i}$ we can choose $a_0, \dots, a_n \in \mathbb{Z}$ with $\gcd(a_0, \dots, a_n) = 1$. If the $\mathbb{Q}$-point lies also in a different chart, then by compatibility of the charts we get (resp. can choose) the same $a_0, \dots, a_n$. Every such chart $U_{x_j,\mathbb{Q}}$ which contains the $\mathbb{Q}$-point induces a ring homomorphism $\mathbb{Z}[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}] \to \mathbb{Z}[a_j^{-1}]$ and thus a morphism $\operatorname{Spec}(\mathbb{Z}[a_j^{-1}]) \to U_{x_j,\mathbb{Z}}$. Since $\gcd(a_0, \ldots, a_n) = 1$ we conclude that the union of all such $\operatorname{Spec}(\mathbb{Z}[a_j^{-1}])$ for $a_j \neq 0$ covers $\mathbb{Z}$. By compatibility the morphisms glue together to give a morphism $\operatorname{Spec}(\mathbb{Z}) \to \mathbb{P}^n_{\mathbb{Z}}$.

(b) As in (a) we see that the number $v(n)$ of elements of $\mathbb{P}^n_{\mathbb{F}_p}(\mathbb{F}_p)$ is equal to the cardinality of the set $(\mathbb{F}_p^{n+1} \setminus \{0\})/\mathbb{F}_p^\times$. The first coordinate of such an element can be zero (in which the others are elements of $\mathbb{P}^{n-1}_{\mathbb{F}_p}(\mathbb{F}_p)$, or 1 (in which case we are on an affine chart). Hence we have $v(n) = v(n-1) + |\mathbb{F}_p^n|$. Recursively we get the formula

$$v(n) = v(0) + |\mathbb{F}_p^1| + \cdots + |\mathbb{F}_p^n| = 1 + p + p^2 + \cdots + p^n = \frac{p^{n+1} - 1}{p - 1}$$

3. Let S be a scheme. For any scheme $X \to S$ over S, the *diagonal* $\Delta_{X/S}$ is the unique morphism $X \to X \times_S X$ such that $p_1 \circ \Delta_{X/S} = p_2 \circ \Delta_{X/S}$ is the identity morphism of X.

(a) Suppose that $X = \operatorname{Spec}(A)$ and $S = \operatorname{Spec}(B)$ are affine. Prove that $\Delta_{X/S}$ is a closed immersion, and prove that the image of $\Delta_{X/S}$ is the closed subscheme of $\operatorname{Spec}(A \otimes_B A)$ defined by the ideal generated by the set of elements of the form $a \otimes 1 - 1 \otimes a$. Let D be the image of $\Delta_{X/S}$; for any scheme T over S, describe the set of T-valued points $D(T)$.

(b) Let Y be any scheme and f_1, f_2 morphisms from X to Y. Suppose that $\Delta_{Y/S}$ is a closed immersion. Prove that the set of $x \in X$ such that $f_1(x) = f_2(x)$ is closed. [Hint: construct a closed immersion, using base change, for which this set is the image.]

(c) Let $S = \operatorname{Spec}(\mathbb{C})$. Let X be the scheme over S defined by gluing $U_1 = \operatorname{Spec}(\mathbb{C}[X_1])$ with $U_2 = \operatorname{Spec}(\mathbb{C}[X_2])$ by identifying $U_{1,2} = \operatorname{Spec}(\mathbb{C}[X_1, X_1^{-1}])$ and $U_{2,1} = \operatorname{Spec}(\mathbb{C}[X_2, X_2^{-1}])$ with the isomorphism $X_1 \mapsto X_2$. Describe $X \times_S X$ and deduce that $\Delta_{X/S}$ is not a closed immersion.

Solution:

(a) The ring homomorphism associated to $\Delta_{X/S}$ is the homomorphism $\varphi : A \otimes_B A \to A$ which sends an elementary tensor $a_1 \otimes a_2$ to $a_1 a_2$, and this homomorphism is surjective. Hence $\Delta_{X/S}$ is a closed immersion. The ideal defining the image is just the kernel of φ. Let I be the ideal generated by all elements of the form $a \otimes 1 - 1 \otimes a$ for $a \in A$. Clearly $I \subset \ker(\varphi)$. Conversely let $\sum_i a_i \otimes b_i$ be an element in $\ker(\varphi)$. By adding zero, we have

$$\sum_i a_i \otimes b_i = \sum_i (a_i \otimes b_i - a_i b_i \otimes 1) = \sum_i (a_i \otimes 1)(1 \otimes b_i - b_i \otimes 1)$$

Which proves that $\sum_i a_i \otimes b_i \in I$.

(b) Consider the base change diagram

$$\begin{array}{ccc} X \times_{(Y \times_S Y)} Y & \longrightarrow & Y \\ \downarrow \varphi & & \downarrow \\ X & \longrightarrow & Y \times_S Y \end{array}$$

where the lower map is the morphism $f_1 \times f_2$. Since $\Delta_{Y/S}$ is a closed immersion, so is φ. The image of φ are precisely all points of x such that $f_1 \times f_2(x)$ lies on the diagonal of Y, hence $f_1(x) = f_2(x)$.

(c) The space $X \times_S X$ is the plane with the two coordinate axis doubled and thus has four origins. Consider the compositions $\varphi_i : \mathbb{A}^1_{\mathbb{C}} \to U_i \to X$ for $i = 1, 2$. We have that $\varphi_1|_{\mathbb{A}^1_{\mathbb{C}} \smallsetminus \{0\}} = \varphi_2|_{\mathbb{A}^1_{\mathbb{C}} \smallsetminus \{0\}}$, but $\varphi_1(\{0\}) \neq \varphi_2(\{0\})$. By the universal property of fiber product, there is an induced morphism $\varphi_1 \times \varphi_2 : \mathbb{A}^1_{\mathbb{C}} \to X \times_S X$. By the above properties of φ_1, φ_2 we see that the preimage of the diagonal by this morphism is $\mathbb{A}^1_{\mathbb{C}} \smallsetminus \{0\}$ which is not closed in $\mathbb{A}^1_{\mathbb{C}}$. We conclude that the diagonal in $X \times_S X$ is not closed.

4. Let K be a field and $n \geqslant 1$. Let X be projective n-space over K and $K(X)$ its function field.

(a) For any prime Weil divisor D on X, defined by the vanishing of an irreducible homogeneous polynomial g, let $\mathcal{L}(D)$ be the sheaf defined by

$$U \mapsto \{f \in K(X) \mid g^{-1}f \text{ is defined on } U\}.$$

Show that $\mathcal{L}(D)$ is invertible, and show that the map $D \mapsto \mathcal{L}(D)$ induces a group homomorphism $\varphi\colon \mathrm{Cl}(X) \to \mathrm{Pic}(X)$.

(b) Let $\mathcal{L}$ be an invertible sheaf on X. Prove that there exists an injective morphism $\mathcal{L} \to \mathcal{K}$, where $\mathcal{K}$ is the constant sheaf $U \mapsto K(X)$.

(c) Deduce that the morphism φ is an isomorphism.

(d) Compute $\varphi^{-1}(\mathcal{O}(1))$.

Solution (sketch):

(a) By definition, on $U = U_{x_i}$ we see that for the polynomial $g_i := g(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i})$ we have $\mathcal{L}(D)|_{U_{x_i}} \cong g_i\mathcal{O}|_{U_{x_i}} \cong \mathcal{O}|_{U_{x_i}}$. Hence the sheaf $\mathcal{L}(D)$ is invertible. We extend $D \mapsto \mathcal{L}(D)$ to all divisors via the tensor product. If $D = \mathrm{div}(f)$ for a rational function $f \in K(X)^\times$ we have the global isomorphism $\mathcal{L}(D) \cong f\mathcal{O}$. Therefore, the map extends to a group homomorphism $\varphi : \mathrm{Cl}(X) \to \mathrm{Pic}(X)$.

(b) Let $\mathcal{L}$ be an invertible sheaf. Consider the tensor product sheaf $\mathcal{L} \otimes_{\mathcal{O}} K(X)$. Locally, this sheaf is isomorphic to the constant sheaf $K(X)$, because $\mathcal{L}$ is locally isomorphic to $\mathcal{O}$. Because X is irreducible, the sheaf $\mathcal{L} \otimes_{\mathcal{O}} K(X)$ is globally isomorphic to the constant sheaf $K(X)$. By the composition $\mathcal{L} \to \mathcal{L} \otimes K(X) \to K(X)$ the sheaf $\mathcal{L}$ is a subsheaf of $K(X)$. The injectivity follows by local injectivity.

(c) Now let (U_i) be a finite covering of X such that $\mathcal{O}|_{U_i} \cong \mathcal{L}|_{U_i}$. We take $f_i \in K(X)$ to be the inverse of the image of 1 under this isomorphism composed with the above morphism to $K(X)$. Now for every prime divisor Y of X let n_Y be the valuation $\nu_Y(f_i)$ for some index i such that $U_i \cap Y \neq \varnothing$. We get a divisor $D(\mathcal{L}) := \sum_Y n_Y Y$. This provides an inverse to the map φ.

(d) We have $\varphi^{-1}(\mathcal{O}(1)) = V(x_0)$.

5. A polynomial $f \in \mathbb{Z}[X_0, X_1, Y_0, Y_1]$ is said to be bi-homogeneous of bi-degree (d_1, d_2) if all monomials that appear with non-zero coefficients are of the form

$$X_0^a X_1^b Y_0^c Y_1^d$$

with $a + b = d_1$, $c + d = d_2$. Let $X = \mathbb{P}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$. Let $U_{i,j}$ be the open subscheme of X isomorphic to $\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$ where X_i and Y_j and invertible.

(a) Let $f \in \mathbb{Z}[X, Y, U, V]$ be a non-constant bi-homogeneous of bi-degree (d_1, d_2). Construct, by suitable gluing, a closed subscheme Y_f of X such that

$$Y \cap U_{i,j} = \operatorname{Spec}(\mathbb{Z}[X_0/X_i, X_1/X_i, Y_0/Y_j, Y_1/Y_j]/f_{i,j})$$

for $0 \leqslant i, j \leqslant 1$, where

$$f_{i,j} = f(X_0/X_i, X_1/X_i, Y_0/Y_j, Y_1/Y_j).$$

(b) Prove that the divisor class group of $\mathbb{P}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Solution (sketch):

(a) We define $Y \cap U_{i,j}$ like above. Since $U_{0,0}, U_{1,0}, U_{0,1}, U_{1,1}$ cover $\mathbb{P}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}$ we can glue those schemes together in the obvious way to get a scheme Y_f.

(b) A prime divisor Y is given by a bi-homogeneous polynomial $f_Y \in \mathbb{Z}[X, Y, U, V]$ similarly to the case for $\mathbb{P}^1$. We define $\deg(Y) \in \mathbb{Z} \times \mathbb{Z}$ to be the bi-degree of f_Y. We extend linearly to a group homomorphism $\operatorname{Cl}(\mathbb{P}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}}) \to \mathbb{Z} \times \mathbb{Z}$. Then part (a) proves the surjectivity of this homomorphism. For the injectivity note that for a divisor $D = \sum n_Y Y$ such that $\sum n_Y \deg(Y) = 0$ we have $\prod f_Y^{n_Y} \in K(\mathbb{P}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}})$ and $D = \operatorname{div}(f)$, hence $D = 0$ in $\operatorname{Cl}(\mathbb{P}^1_{\mathbb{Z}} \times \mathbb{P}^1_{\mathbb{Z}})$.