Exercise Sheet 2

1) Let G be a compact connected Lie group and let

$$G^* = \{(g, g) \in G \times G : g \in G\} < G$$

denote the diagonal subgroup.

a) Show that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair, and the coset space $G \times G/G^*$ is diffeomorphic to G.

b) Using the above, explain how any compact connected Lie group G can be regarded as a Riemannian globally symmetric space.

c) Let $\mathfrak{g}$ denote the Lie algebra of G. Show that the exponential map from $\mathfrak{g}$ into the Lie group G coincides with the exponential map from $\mathfrak{g}$ into the Riemannian *globally symmetric space* G.

2) A compact semisimple Lie group G has a bi-invariant Riemannian structure Q such that Q_e is the negative of the Killing form of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$. If G is considered as a symmetric space $G \times G/G^*$ as in the above exercise, it acquires a bi-invariant Riemannian structure Q^* from the Killing form of $\mathfrak{g}\times\mathfrak{g}$. Show that $Q = 2Q^*$.

3) Show that any two complete simply connected Riemannian manifolds of the same dimension and of the same constant sectional curvature are isometric.

Hint: It may be useful to first prove the following fact:

Let V and W be Riemannian manifolds, let V be complete, and let $\phi : V \to W$ be a surjective differentiable map. Assume that $d_v\phi$ is an isometry for each $v \in V$. Then (V, ϕ) is a covering space for W.

4) Let M be a Riemannian globally symmetric space and let ω be a differential form on M invariant under $\text{Isom}(M)^\circ$. Prove that $d\omega = 0$.