Exercise Sheet 4

1) Let $\mathfrak{l}_0$ be a Lie algebra over $\mathbb{R}$ and let $\mathfrak{l}$ be the complexification of $\mathfrak{l}_0$. Let K_0, K and $K^{\mathbb{R}}$ denote the Killing forms of the Lie algebras $\mathfrak{l}$, $\mathfrak{l}_0$ and $\mathfrak{l}^{\mathbb{R}}$, respectively. Show that:

(a) $K_0(X, Y) = K(X, Y)$ for all $X, Y \in \mathfrak{l}_0$;

(b) $K^{\mathbb{R}}(X, Y) = 2 \cdot \mathrm{Re}(K(X, Y))$ for all $X, Y \in \mathfrak{l}^{\mathbb{R}}$.

2) Let $(\mathfrak{l}, \mathfrak{s})$ be an orthogonal symmetric Lie algebra with $\mathfrak{l}$ semisimple. Show that:

(a) $\mathfrak{u}$ equals its normalizer in $\mathfrak{l}$;

(b) if $\mathfrak{u}$ contains no ideal in $\mathfrak{l}$ then $[\mathfrak{e}, \mathfrak{e}] = \mathfrak{u}$.

3) $\mathfrak{so}(1, 3) \cong \mathfrak{sl}(2, \mathbb{C})^{\mathbb{R}}$.

Exhibit an explicit isomorphism between $\mathfrak{so}(1, 3)$ and $\mathfrak{sl}(2, \mathbb{C})$.

<u>Hint:</u> Consider the the vector space V of 2×2-skew-Hermitian matrices and endow it with the quadratic form $q(v) := \det(v)$. Now, let $\mathrm{SL}(2, \mathbb{C})$ act on V via $g.v := gv\bar{g}^t$.

4) Show that the symmetric spaces $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathbb{H}^n \cong \mathrm{SO}(1, n)^\circ/\mathrm{SO}(n)$ are dual to each other.