Exercise Sheet 5

1) Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. A Cartan decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{p} = \{X \in \mathfrak{sl}(n, \mathbb{R}) : X = X^t\}$ and $\mathfrak{k} = \{X \in \mathfrak{sl}(n, \mathbb{R}) : X = -X^t\}$. Define

$$\mathfrak{a} = \left\{ \operatorname{diag}(t_1, \ldots, t_n) : t_j \in \mathbb{R}, \sum_{j=1}^{n} t_j = 0 \right\}.$$

(a) Prove that $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$.

(b) Prove (without appealing to the general theorem) any maximal abelian subspace of $\mathfrak{p}$ is of the form $S\mathfrak{a}S^{-1}$ where $S \in SO(n)$.

(c) Show that $X \in \mathfrak{p}$ is a regular element if and only if all of its eigenvalues are distinct.

2) Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$. Recall that a Cartan decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A = A^t, B = B^t \right\}$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A = -A^t, B = B^t \right\}.$$

(a) Define

$$\mathfrak{a} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} : A = \operatorname{diag}(t_1, \ldots, t_n). \right\}.$$

Prove that A is a maximal abelian subspace of $\mathfrak{p}$.

(b) Show that $X \in \mathfrak{p}$ is a regular element if and only if all of its eigenvalues are distinct and non-zero.

3) Let

$$\mathbb{H}_n := \{Z \in \mathbb{C}^{n\times n} : Z = Z^t,\ \operatorname{Im}(Z) \text{ is positive-definite}\}.$$

Find an explicit isomorphism between $\mathbb{H}_n$ and $\operatorname{Sp}(2n, \mathbb{R})/(\operatorname{SO}(2n) \cap \operatorname{Sp}(2n, \mathbb{R}))$. Use this and the previous exercise to construct a maximal flat of $\mathbb{H}_n$.

<u>Hint:</u> : Consider the map

$$\phi : \operatorname{Sp}(2n, \mathbb{R}) \to \mathbb{H}_n,$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto (Ai + B) \cdot (Ci + D)^{-1}.$$

4) Let $V = \mathbb{C}[X, Y]$ be the vector space of polynomials in two variables. Let V_m denote the vector subspace of all homogeneous polynomials of degree m. This has a basis given by the monomials $X^m, X^{m-1}Y, \ldots, Y^m$. We turn this vector subspace into a module for $\mathfrak{sl}(2, \mathbb{C})$ by defining a Lie algebra homomorphism $\phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(V_m)$ in the following way

$$\phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = X\frac{\partial}{\partial Y}, \quad \phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = Y\frac{\partial}{\partial X}, \quad \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Show that this defines an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$.