Lie Groups II (Prof. Iozzi) - Solution to Set 2

1. Let G be a compact connected Lie group. Let G^* denote the diagonal of $G \times G$.

(1) Show that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair, and the coset space $G \times G/G^*$ is diffeomorphic to G.

(2) Using the above, explain how any compact connected Lie group G can be regarded as a Riemannian globally symmetric space.

(3) Let $\mathfrak{g}$ denote the Lie algebra of G. Show that the exponential map from $\mathfrak{g}$ into the Lie group G coincides with the exponential map from $\mathfrak{g}$ into the *Riemannian globally symmetric space* G.

Solution. Consider the mapping $\sigma : (g_1, g_2) \mapsto (g_2, g_1)$. This is an involutive automorphism of the product group $G \times G$. The fixed set of σ is precisely the diagonal G^*. It follows that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair. The coset space $G \times G/G^*$ is diffeomorphic to G under the mapping $[(g_1, g_2)] \mapsto \pi(g_1, g_2) = g_1 g_2^{-1}$.

Next, by Proposition 3.4 from Helgason, Ch. IV, we see that G is a Riemannian globally symmetric space in each bi-invariant Riemannian structure; note here that a Riemannian structure on $G \times G/G^*$ is $G \times G-$invariant iff the corresponding Riemannian structure on G is bi-translation invariant.

Finally, note that the product algebra $\mathfrak{g} \times \mathfrak{g}$ is the Lie algebra of $G \times G$. Let $\exp^*$ denote the exponential mapping of $\mathfrak{g} \times \mathfrak{g}$ into $G \times G$, exp denote the exponential mapping of $\mathfrak{g}$ into G (considered as a Lie group), and Exp denote the Exponential mapping of $\mathfrak{g}$ into G (considered as a Riemannian globally symmetric space). We want to show that $\exp X = \mathrm{Exp} X$ for all $X \in \mathfrak{g}$. Using $d\pi(X, Y) = X - Y$, we deduce that $\pi(\exp^*(X, -X)) = \mathrm{Exp}(d\pi(X, -X))$. Hence $\exp X \cdot (\exp(-X))^{-1} = \mathrm{Exp}(2X)$ and this implies that $\exp X = \mathrm{Exp} X$.

2. A compact semisimple Lie group G has a bi-invariant Riemannian structure Q such that Q_e is the negative of the killing form of the Lie algebra $\mathfrak{g}$ of G. If G is considered as a symmetric space $G \times G/G^*$ as in the above exercise, it acquires a bi-invariant Riemannian structure Q^* from the killing form of $\mathfrak{g} \times \mathfrak{g}$. Show that $Q = 2Q^*$.

Solution. Let π and σ be as in the above solution. The map $d\pi$ maps the -1 eigenspace of $d\sigma$ onto $\mathfrak{g}$ as follows: $d\pi(X, -X) = 2X$. Using this, we can check that

$$2B_{\mathfrak{g} \times \mathfrak{g}}((X, -X), (X, -X)) = B_{\mathfrak{g}}(2X, 2X),$$

which is equivalent to $Q = 2Q^*$.

3. Show that any two complete simply connected Riemannian manifolds of the same dimension and of the same constant sectional curvature are isometric.

Hint. It may be helpful to first prove the following fact:

Let V and W be Riemannian manifolds with V complete, and let $\phi : V \to W$ be a surjective differentiable map. Assume that $d\phi_v$ is an isometry for each $v \in V$. Then (V, ϕ) is a covering space for W.

Solution. The helpful fact given in the question is Lemma 13.4 of Helgason, Ch I.

Lemma 1.2 of Helgason, Chapter IV gives an isometry between two normal neighbor-hoods of V and W. Using this we deduce that $d\phi_v$ is an isometry for each $v \in V$. Now the helpful fact tells us that (V, ϕ) is a covering space for W. Since both spaces are simply connected, it follows that V and W are isometric.

4. Let M be a Riemannian globally symmetric space, ω a differential form on M invariant under each member of $I_0(M)$. Prove that $d\omega = 0$.

Solution. Since ω is invariant, it follows that $s_m\omega$ is also invariant for each $m \in M$. On the other hand, $s_m\omega = (-1)^p\omega$ if ω is a p-form. It follows that

$$d\omega = (-1)^p d(s_m\omega) = (-1)^p s_m(d\omega) = (-1)^{2p+1} d\omega,$$

and hence $d\omega = 0$.