

## Solution Exercise Sheet 3

### 1) Details on $\mathrm{SO}(1, n)^\circ / \mathrm{SO}(n)$ .

Consider  $G = \mathrm{SO}(1, n)^\circ$  with the involutive Lie group automorphism

$$\sigma : G \rightarrow G, g \mapsto J_n g J_n$$

where

$$J_n = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \in \mathrm{SO}(1, n).$$

Further let

$$K = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(n) \end{pmatrix} \cong \mathrm{SO}(n).$$

We have seen in the lecture that  $(G, K, \sigma)$  is a Riemannian symmetric pair and that  $G/K$  is isometric to  $\mathbb{H}^n$ . The objective of this exercise is to verify the formulas that we have used in the lecture to study  $G/K$ .

(a) Show that  $\Theta = d\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$  takes the form

$$\Theta(X) = \begin{pmatrix} 0 & -x^t \\ -x & D \end{pmatrix}$$

for all

$$X = \begin{pmatrix} 0 & x^t \\ x & D \end{pmatrix} \in \mathfrak{g} = \mathfrak{so}(1, n).$$

Deduce that

$$\mathfrak{p} = E_{-1}(\Theta) = \left\{ \begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix} : x \in \mathbb{R}^n \right\}, \mathfrak{k} = E_1(\Theta) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} : D \in \mathfrak{so}(n) \right\} \cong \mathfrak{so}(n),$$

*Solution:* We compute

$$\begin{aligned} \Theta(X) &= J_n X J_n = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & x^t \\ x & D \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \\ &= \begin{pmatrix} 0 & -x^t \\ x & D \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} 0 & -x^t \\ -x & D \end{pmatrix} \end{aligned}$$

for all  $X = \begin{pmatrix} 0 & x^t \\ x & D \end{pmatrix} \in \mathfrak{so}(1, n)$ . Thus  $\Theta(X) = -X$  implies  $D = 0$ , and likewise  $\Theta(X) = X$  implies  $x = 0$ .

(b) Let  $\pi : G \rightarrow G/K$  denote the usual quotient map and set  $\bar{X} := d_e \pi(X) \in T_o(G/K)$  for all  $X \in \mathfrak{g}$ . Further let  $\langle X, Y \rangle := \frac{1}{2} \mathrm{tr}(XY)$  for all  $X, Y \in \mathfrak{p}$  as in the lecture.

Show that

$$R_o(\bar{X}, \bar{Y})\bar{Z} = \langle X, Z \rangle \bar{Y} - \langle Y, Z \rangle \bar{X}$$

for all  $X, Y, Z \in \mathfrak{p}$ . Deduce that  $G/K$  has constant sectional curvature  $-1$ .

*Solution:* Let  $X = \begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & y^t \\ y & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & z^t \\ z & 0 \end{pmatrix} \in \mathfrak{p}$ . Note that

$$\langle X, Y \rangle = \frac{1}{2} \operatorname{tr} \left( \begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix} \begin{pmatrix} 0 & y^t \\ y & 0 \end{pmatrix} \right) = \frac{1}{2} \operatorname{tr} \begin{pmatrix} x^t y & 0 \\ 0 & x y^t \end{pmatrix} = \langle x, y \rangle,$$

where the latter is to be understood as the Euclidean inner product of the vectors  $x, y \in \mathbb{R}^n$ .

By the formula from the lecture for the Riemannian curvature tensor we have that

$$R_o(\bar{X}, \bar{Y})\bar{Z} = -\overline{[[X, Y], Z]}.$$

First, we compute

$$\begin{aligned} [X, Y] &= \begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix} \begin{pmatrix} 0 & y^t \\ y & 0 \end{pmatrix} - \begin{pmatrix} 0 & y^t \\ y & 0 \end{pmatrix} \begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix} \\ &= \begin{pmatrix} x^t y & 0 \\ 0 & x y^t \end{pmatrix} - \begin{pmatrix} y^t x & 0 \\ 0 & y x^t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & x y^t - y x^t \end{pmatrix}, \end{aligned}$$

and then

$$\begin{aligned} [[X, Y], Z] &= \begin{pmatrix} 0 & 0 \\ 0 & x y^t - y x^t \end{pmatrix} \begin{pmatrix} 0 & z^t \\ z & 0 \end{pmatrix} - \begin{pmatrix} 0 & z^t \\ z & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & x y^t - y x^t \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ (x y^t - y x^t) z & 0 \end{pmatrix} - \begin{pmatrix} 0 & z^t (x y^t - y x^t) \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \langle y, z \rangle x^t - \langle x, z \rangle y^t \\ \langle y, z \rangle x - \langle x, z \rangle y & 0 \end{pmatrix} \\ &= \langle Y, Z \rangle X - \langle X, Z \rangle Y. \end{aligned}$$

That implies our claim.

As for the sectional curvature let  $V \subset \mathfrak{p}$  be a two-dimensional linear subspace and choose  $X, Y \in \mathfrak{p}$  to be an orthonormal basis of  $V$ . Then

$$\kappa_o(\bar{V}) = R_o(\bar{X}, \bar{Y})\bar{Y}, \bar{X} = \langle R_o(\bar{X}, \bar{Y})\bar{Y}, \bar{X} \rangle = \langle \langle X, Y \rangle \bar{Y} - \langle Y, Y \rangle \bar{X}, \bar{X} \rangle = -1.$$

(c) Compute that

$$\exp \left( t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

for all  $t \in \mathbb{R}$ .

*Solution:* Note that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{2k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{2k+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for all  $k \in \mathbb{N}$ . Thus

$$\begin{aligned}
\exp\left(t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n \\
&= \sum_{k=0}^{\infty} \frac{t^{2k}}{2k!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{2k} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{2k+1} \\
&= \cosh t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}
\end{aligned}$$

for all  $t \in \mathbb{R}$ .

## 2) Closed adjoint subgroups of $\mathrm{SL}_n(\mathbb{R})$ and their symmetric spaces.

Consider the Riemannian symmetric pair  $(G, K, \sigma)$  where  $G = \mathrm{SL}_n(\mathbb{R})$ ,  $K = \mathrm{SO}(n, \mathbb{R})$  and  $\sigma : \mathrm{SL}_n(\mathbb{R}) \rightarrow \mathrm{SL}_n(\mathbb{R})$ ,  $g \mapsto (g^{-1})^t$ . Further let  $H \leq G$  be a closed, connected subgroup that is adjoint, i.e. it is closed under transposition  $h \mapsto h^t$ .

- (a) Show that  $(H, H \cap K, \sigma|_H)$  is again a Riemannian symmetric pair.

*Solution:* Note that  $H = \sigma(\sigma(H)) \subseteq \sigma(H)$  since  $\sigma : G \rightarrow G$  is an involutive Lie group automorphism. Because  $H \leq G$  is a closed subgroup and hence an embedded submanifold  $\sigma$  restricts to an involutive Lie group automorphism  $\sigma|_H : H \rightarrow H$ .

Let us check that  $(H^\sigma)^\circ \subseteq H \cap K \subseteq H^\sigma$ . Concerning the first inclusion note that  $(H^\sigma)^\circ \subseteq H^\sigma \subseteq G^\sigma$  whence  $(H^\sigma)^\circ \subseteq (G^\sigma)^\circ$ . Then the first inclusion follows from  $(H^\sigma)^\circ \subseteq H$  and  $(G^\sigma)^\circ \subseteq K$ . The second inclusion follows easily from the fact that  $H^\sigma = H \cap G^\sigma$  and  $K \subseteq G^\sigma$ .

Finally,  $K = \mathrm{SO}(n, \mathbb{R})$  is compact whence the intersection  $H \cap K$  is also compact as is the image  $\mathrm{Ad}_H(H \cap K) \leq \mathrm{GL}(\mathfrak{h})$ .

- (b) Show that  $i : H \hookrightarrow G$  descends to a smooth embedding  $\phi : H/H \cap K \hookrightarrow G/K$  such that its image is a totally geodesic submanifold of  $G/K$ .

*Solution:* Denote by  $\pi : G \rightarrow G/K$  and  $\pi' : H \rightarrow H/H \cap K$  the usual quotient maps. Define

$$\phi : H/H \cap K \rightarrow G/K, \pi'(h) \mapsto \pi(i(h)).$$

That map is well-defined because

$$\begin{aligned}
\pi'(h_1) = \pi'(h_2) &\iff h_2^{-1}h_1 \in H \cap K \\
&\iff i(h_2^{-1}h_1) \in K \\
&\iff \pi(i(h_1)) = \pi(i(h_2))
\end{aligned}$$

for all  $h_1, h_2 \in H$ . That argument also shows that  $\phi$  is injective whence it is a bijection onto its image  $N := \mathrm{im} \phi$ . We obtain the following commutative diagram.

$$\begin{array}{ccc}
H & \xrightarrow{i} & G \\
\downarrow \pi' & & \downarrow \pi \\
H/H \cap K & \xrightarrow{\phi} & N \subseteq G/K
\end{array}$$

Clearly,  $\phi$  is continuous by the universal property of the quotient topology. We will now show that  $\phi : H/H \cap K \rightarrow N$  is proper, i.e. preimages of compact sets are compact. That will prove that  $\phi$  is actually open onto its image because proper continuous maps are closed and continuous closed bijections are open. Let  $C \subseteq N \subset G/K$  be compact. Because  $G$  is locally compact there is a compact set  $C' \subseteq G$  such that  $\pi(C') = C$  and

$$\pi^{-1}(C) = \pi^{-1}(\pi(C')) = \bigcup_{k \in K} C'k$$

is compact, i.e.  $\pi$  is proper. Further  $i^{-1}(\pi^{-1}(C)) = H \cap \pi^{-1}(C)$  is compact since  $H$  is closed. Finally,  $\phi^{-1}(C) = \pi'(i^{-1}(\pi^{-1}(C)))$  is compact since  $\pi'$  is continuous. Therefore,  $\phi : H/H \cap K \rightarrow N$  is a homeomorphism.

Note that the smooth structure on  $H/H \cap K$  is such that  $\pi'$  is a smooth submersion whence  $\phi : H/H \cap K \rightarrow G/K$  is smooth because  $\pi \circ i : H \rightarrow G/K$  is smooth. Also,  $\phi$  is equivariant with respect to the respective  $H$ -actions on  $H/H \cap K$  and  $N$ . Because  $H$  acts transitively on  $H/H \cap K$  it is easy to see that  $\phi$  has constant rank. By the global rank theorem [2, Theorem 4.14]  $\phi : H/H \cap K \rightarrow N$  is a smooth immersion. This shows that  $\phi$  is a smooth embedding.

In order to check that  $N$  is a totally geodesic submanifold we will show that its tangent space amounts to a Lie triple system  $\mathfrak{n} \subseteq \mathfrak{p}$ . Let  $\mathfrak{h} = \text{Lie}(H)$  and  $\mathfrak{g} = \text{Lie}(G)$  with Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . Define  $\Theta|_H := d\sigma|_H : \mathfrak{h} \rightarrow \mathfrak{h}$ . It is easy to check that the corresponding Cartan decomposition is just  $\mathfrak{h} = \mathfrak{k}' \oplus \mathfrak{p}'$  with  $\mathfrak{k}' := \mathfrak{k} \cap \mathfrak{h}$  and  $\mathfrak{p}' := \mathfrak{p} \cap \mathfrak{h}$ . As we know  $d_e\pi' : \mathfrak{p}' \rightarrow T_oH/H \cap K$  is an isomorphism as is  $d_o\phi : T_oH/H \cap K \rightarrow T_oN$ . By commutativity of the above diagram  $T_oN = d_e\pi(\mathfrak{n})$  where  $\mathfrak{n} = di(\mathfrak{p}') \subseteq \mathfrak{p}$ . The subspace  $\mathfrak{n}$  is a Lie triple system since  $di : \mathfrak{h} \rightarrow \mathfrak{g}$  is a Lie algebra homomorphism

$$[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] = [[di(\mathfrak{p}'), di(\mathfrak{p}')], di(\mathfrak{p}')] = di([[p', p'], p']) \subseteq di([\mathfrak{k}', p']) \subseteq di(\mathfrak{p}') = \mathfrak{n}.$$

Thus  $N$  is a totally geodesic submanifold.

### 3) The symplectic group $\text{Sp}(2n, \mathbb{R})$ .

Let  $H = \text{Sp}(2n, \mathbb{R}) = \{g \in \text{GL}_{2n}(\mathbb{R}) : g^t J g = J\}$  be the symplectic group, where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

- (a) Show that  $\text{Sp}(2n, \mathbb{R}) \leq \text{SL}(2n, \mathbb{R}) =: G$  is a closed connected *adjoint* subgroup of  $G$ .

What can we deduce from exercise 2 about  $(H, H \cap K, \sigma|_H)$ ?

*Solution:* We will only show that  $\text{Sp}(2n, \mathbb{R})$  is adjoint. Let  $g \in \text{Sp}(2n, \mathbb{R})$ . Note that  $J = -J$ . Then

$$g^t J g = J \implies g^t = -J g^{-1} J$$

and thus

$$g J g^t = g J (-J g^{-1} J) = g g^{-1} J = J$$

whence  $g^t \in \text{Sp}(2n, \mathbb{R})$ .

Now set  $K' := \text{Sp}(2n, \mathbb{R}) \cap \text{SO}(2n, \mathbb{R})$ . By exercise 2  $H/K'$  is again a symmetric space and the inclusion  $H \hookrightarrow \text{SL}(2n, \mathbb{R})$  descends to a smooth embedding with image a totally geodesic submanifold of  $\text{SL}(2n, \mathbb{R})/\text{SO}(2n, \mathbb{R})$ .

- (b) Denote by  $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$  the standard symplectic form given by  $\omega(x, y) = x^t J y$ . Show that  $B : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}, (x, y) \mapsto \omega(Jx, y)$  is a symmetric positive definite bilinear form.

*Solution:* Let  $x, y \in \mathbb{R}^{2n}$ . Then

$$B(x, y) = \omega(Jx, y) = (Jx)^t J y = -x^t J^2 y = x^t y = \langle x, y \rangle.$$

- (c) An endomorphism  $M \in \text{End}(\mathbb{R}^{2n})$  is called a complex structure if  $M^2 = -\text{id}$ . We say that  $M$  is  $\omega$ -compatible if  $(x, y) \mapsto \omega(Mx, y)$  is a symmetric positive definite bilinear form. Denote the set of all  $\omega$ -compatible complex structures by  $S_{2n}$ .

Show that  $H = \text{Sp}(2n, \mathbb{R})$  acts on  $S_{2n}$  via conjugation and deduce that there is a bijection  $S_{2n} \cong H/H \cap K$ .

*Solution:* Let  $M \in S_{2n}$  and  $g \in \text{Sp}(2n, \mathbb{R})$ . Then

$$(gMg^{-1})^2 = gM^2g^{-1} = -gg^{-1} = -\text{id}.$$

If we denote by  $B_M(x, y) = \omega(Mx, y)$  the associated symmetric positive definite bilinear form then

$$\begin{aligned} B_{gMg^{-1}}(x, y) &= \omega(gMg^{-1}x, y) = \omega(Mg^{-1}x, g^{-1}y) \\ &= B_M(g^{-1}x, g^{-1}y) = g_* B_M(x, y) \end{aligned}$$

for all  $x, y \in \mathbb{R}^{2n}$ , which is again a symmetric positive definite bilinear form.

Therefore,  $\text{Sp}(2n, \mathbb{R})$  acts via conjugation on  $S_{2n}$ . By b)  $J$  is in  $S_{2n}$ . Its stabilizer is  $\text{stab}_{\text{Sp}(2n, \mathbb{R})}(J) = \text{Sp}(2n, \mathbb{R}) \cap \text{SO}(2n, \mathbb{R}) = \text{Sp}(2n, \mathbb{R}) \cap O(2n, \mathbb{R})$ . Indeed

$$\begin{aligned} g \in \text{stab}_{\text{Sp}(2n, \mathbb{R})}(J) &\iff gJg^{-1} = J \\ &\iff \omega^{\text{non-deg}}(gJg^{-1}x, y) = \omega(Jx, y) \quad \forall x, y \in \mathbb{R}^{2n} \\ &\iff \omega(Jg^{-1}x, g^{-1}y) = \omega(Jx, y) \quad \forall x, y \in \mathbb{R}^{2n} \\ &\iff \omega^{(J \cdot, \cdot) = \langle \cdot, \cdot \rangle}(g^{-1}x, g^{-1}y) = \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^{2n} \\ &\iff g \in \text{Sp}(2n, \mathbb{R}) \cap O(2n, \mathbb{R}). \end{aligned}$$

Using symplectic linear algebra one can show that  $M$  is  $\omega$ -compatible if and only if there is a symplectic basis for  $\mathbb{R}^{2n}$  of the form

$$e'_1, \dots, e'_n, f'_1, \dots, f'_n = Me'_1, \dots, f'_n = Me'_n,$$

i.e.  $\omega(e'_i, e'_j) = 0, \omega(f'_i, f'_j) = 0, \omega(e'_i, f'_j) = \delta_{ij}$  (see [1, Ex. 3, p. 73]). Now define

$$\begin{aligned} g : \mathbb{R}^{2n} &\rightarrow \mathbb{R}^{2n}, \\ e_i &\mapsto e'_i \\ f_j &\mapsto f'_j \end{aligned}$$

by linear extension where  $\{e_i, f_j = J e_j\}$  denotes the standard symplectic basis of  $\mathbb{R}^{2n}$ . Since both  $\{e_i, f_j\}$  and  $\{e'_i, f'_j\}$  are symplectic bases the map  $g$  is a symplectomorphism of  $(\mathbb{R}^{2n}, \omega)$  onto itself, i.e.  $g \in \text{Sp}(2n, \mathbb{R})$ . Further

$$Mg(e_i) = Me'_i = f'_i = g(f_i) = gJ(e_i)$$

and

$$Mg(f_j) = Mf'_j = -e'_j = -g(e_j) = g(Jf_j),$$

so that  $Mg = gJ$ , or equivalently  $M = gJg^{-1}$ . That shows that the action is transitive.

Therefore

$$\mathrm{Sp}(2n, \mathbb{R}) / \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{SO}(2n, \mathbb{R}) \cong S_{2n}.$$

## References

- [1] Ana Cannas Da Silva and F Takens. *Lectures on symplectic geometry*, volume 3575. Springer, 2001. Available at <https://people.math.ethz.ch/~acannas/Papers/lsg.pdf>.
- [2] John M Lee. Introduction to smooth manifolds. number 218 in graduate texts in mathematics, 2003.