

## Solution Exercise Sheet 4

- 1) Let  $\mathfrak{l}_0$  be a Lie algebra over  $\mathbb{R}$  and let  $\mathfrak{l}$  be the complexification of  $\mathfrak{l}_0$ . Let  $K_0, K$  and  $K^{\mathbb{R}}$  denote the Killing forms of the Lie algebras  $\mathfrak{l}, \mathfrak{l}_0$  and  $\mathfrak{l}^{\mathbb{R}}$ , respectively. Show that:

- (a)  $K_0(X, Y) = K(X, Y)$  for all  $X, Y \in \mathfrak{l}_0$ ;
- (b)  $K^{\mathbb{R}}(X, Y) = 2 \cdot \operatorname{Re}(K(X, Y))$  for all  $X, Y \in \mathfrak{l}^{\mathbb{R}}$ .

*Solution:* The first relation is obvious. For the second let  $\mathcal{B} := \{X_i : i = 1, \dots, n\}$  be a basis of  $\mathfrak{l}$ . Let  $X, Y \in \mathfrak{l}$ . Then we may write

$$\operatorname{ad}(X) \operatorname{ad}(Y)(X_i) = \sum_{j=1}^n \alpha_{ij} \cdot X_j, \quad i = 1, \dots, n, \quad (1)$$

for some complex numbers  $\alpha_{ij} = \beta_{ij} + i \cdot \gamma_{ij} \in \mathbb{C}$ . Denote by  $A, B, C$  the  $n \times n$ -matrices with entries  $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ , respectively. Then  $A$  is the matrix representation of  $\operatorname{ad}(X) \operatorname{ad}(Y)$  with respect to the basis  $\mathcal{B}$

$$M_{\mathcal{B}}(\operatorname{ad}(X) \operatorname{ad}(Y)) = A = B + iC$$

and  $B, C$  are the real, imaginary parts of  $A$ . Now, consider the basis  $\mathcal{C} = \{X_1, \dots, X_n, iX_1, \dots, iX_n\}$  of  $\mathfrak{l}^{\mathbb{R}}$ . Then

$$\operatorname{ad}(X) \operatorname{ad}(Y)(iX_i) = \sum_{j=1}^n -\gamma_{ij} \cdot X_j + \sum_{j=1}^n \beta_{ij} \cdot (iX_j), \quad i = 1, \dots, n, \quad (2)$$

and with (1) we obtain that the matrix representation of  $\operatorname{ad}(X) \operatorname{ad}(Y)$  with respect to the basis  $\mathcal{C}$  is given by

$$A' := M_{\mathcal{C}}(\operatorname{ad}(X) \operatorname{ad}(Y)) = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

Thus

$$2 \operatorname{Re} K(X, Y) = 2 \operatorname{Re}(\operatorname{tr} A) = 2B = \operatorname{tr} A' = K^{\mathbb{R}}(X, Y).$$

- 2) Let  $(\mathfrak{l}, \mathfrak{s})$  be an orthogonal symmetric Lie algebra with  $\mathfrak{l}$  semisimple. Show that:

- (a)  $\mathfrak{u}$  equals its normalizer in  $\mathfrak{l}$ ;
- (b) if  $\mathfrak{u}$  contains no ideal in  $\mathfrak{l}$  then  $[\mathfrak{e}, \mathfrak{e}] = \mathfrak{u}$ .

*Solution:*

To (a): Decompose  $\mathfrak{l} = \mathfrak{u} + \mathfrak{e}$ . It will suffice to show that if  $X \in \mathfrak{e}$  is in the normalizer of  $\mathfrak{u}$  then  $X = 0$ .

To that end note that  $[X, \mathfrak{u}] \subseteq [\mathfrak{e}, \mathfrak{u}] \subseteq \mathfrak{e}$  and also  $[X, \mathfrak{u}] \subseteq \mathfrak{u}$  since  $X$  is in the normalizer of  $\mathfrak{u}$ . Thus  $[X, \mathfrak{u}] \in \mathfrak{u} \cap \mathfrak{e} = 0$ , i.e.  $X$  commutes with every element of  $\mathfrak{u}$ . Let us further decompose  $X = X_0 + X_- + X_+ \in \mathfrak{e}_0 + \mathfrak{e}_- + \mathfrak{e}_+$ . Since  $\mathfrak{e}_0, \mathfrak{e}_-$  and  $\mathfrak{e}_+$  are invariant under  $\operatorname{ad}(\mathfrak{u})$  we have

$$0 = [X, \mathfrak{u}] = \underbrace{\operatorname{ad}(\mathfrak{u})(X_0)}_{\subseteq \mathfrak{e}_0} + \underbrace{\operatorname{ad}(\mathfrak{u})(X_+)}_{\subseteq \mathfrak{e}_+} + \underbrace{\operatorname{ad}(\mathfrak{u})(X_-)}_{\subseteq \mathfrak{e}_-},$$

such that  $[X_0, u] = [X_-, u] = [X_+, u] = 0$ . It follows that

$$\text{ad}(X_+) \text{ad}(X_+)u = 0 \quad \text{and} \quad \text{ad}(X_+) \text{ad}(X_+)e = [X_+, \underbrace{[X_+, e]}_{\subseteq u}] = 0,$$

whence  $\text{ad}(X_+)^2 \equiv 0$ , and  $\text{ad}(X_-)^2 \equiv 0$ , analogously. Thus

$$B(X_+, X_+) = \text{tr ad}(X_+)^2 = 0 = B(X_-, X_-)$$

and  $X_+ = X_- = 0$ , since  $B|_{\mathfrak{e}_+ \times \mathfrak{e}_+} \gg 0$  and  $B|_{\mathfrak{e}_- \times \mathfrak{e}_-} \ll 0$ . Therefore  $X = X_0 \in \mathfrak{e}_0$ . However,  $\mathfrak{l}$  is semisimple and  $\mathfrak{e}_0$  is an abelian ideal whence  $\mathfrak{e}_0 = 0$ .

Remark: The above argument also shows that we have in every orthogonal symmetric Lie algebra  $\mathfrak{l}$  (not necessarily semisimple):

*If  $X \in \mathfrak{e}$  commutes with every element in  $\mathfrak{u}$  then  $X \in \mathfrak{e}_0$  (see [1, Chapter V, Corollary 1.7]).*

To (b): We decompose  $\mathfrak{e} = \mathfrak{e}_0 + \mathfrak{e}_- + \mathfrak{e}_+$  and  $\mathfrak{u} = \mathfrak{u}_0 + \mathfrak{u}_- + \mathfrak{u}_+$ . By our assumption  $\mathfrak{e}_0 = 0$  such that  $[\mathfrak{u}_0, \mathfrak{e}] = [\mathfrak{u}_0, \mathfrak{e}_+] + [\mathfrak{u}_0, \mathfrak{e}_-] = 0$  whence  $\mathfrak{u}_0$  is an ideal in  $\mathfrak{l}$  so that  $\mathfrak{u}_0 = 0$  by our hypothesis. Finally,

$$[\mathfrak{e}, \mathfrak{e}] = [\mathfrak{e}_+, \mathfrak{e}_+] + [\mathfrak{e}_-, \mathfrak{e}_-] = \mathfrak{u}_+ + \mathfrak{u}_- = \mathfrak{u}$$

because  $[\mathfrak{e}_+, \mathfrak{e}_-] = 0$ .

### 3) $\mathfrak{so}(1, 3) \cong \mathfrak{sl}(2, \mathbb{C})^{\mathbb{R}}$ .

Exhibit an explicit isomorphism between  $\mathfrak{so}(1, 3)$  and  $\mathfrak{sl}(2, \mathbb{C})$ .

Hint: Consider the the vector space  $V$  of  $2 \times 2$ -skew-Hermitian matrices and endow it with the quadratic form  $q(v) := \det(v)$ . Now, let  $\text{SL}(2, \mathbb{C})$  act on  $V$  via  $g.v := gv\bar{g}^t$ .

Solution: Every element  $v \in V = \{v \in \mathbb{C}^{2 \times 2} : \bar{v}^t = -v\}$  can be written as

$$v = \begin{pmatrix} i(x_1 - x_3) & -x_2 + ix_4 \\ x_2 + ix_4 & i(x_1 + x_3) \end{pmatrix}$$

where  $x_1, x_2, x_3, x_4 \in \mathbb{R}$ . We compute

$$q(v) = \det(v) = -x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

It is readily verified that the given action of  $\text{SL}(2, \mathbb{C})$  on  $V$  is well-defined and preserves  $q$ . Indeed,

$$q(g.v) = \det(gv\bar{g}^t) = \det(g) \det(v) \det(\bar{g})^t = \det(v)$$

for every  $g \in \text{SL}(2, \mathbb{C})$  and every  $v \in V$ .

Thus we obtain a Lie group homomorphism  $\phi : \text{SL}(2, \mathbb{C}) \rightarrow \text{SO}(1, 3)^\circ$ ,  $\phi(g)(v) = g.v$ . It is easy to check that  $\{\pm I\} \subseteq \ker \phi$ . Further, if  $\phi(g) = I$  then in particular

$$\begin{aligned} g \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \bar{g}^t &= \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \\ g \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{g}^t &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ g \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \bar{g}^t &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \end{aligned}$$

and it is elementary to deduce that  $g = \pm I$ . Hence,  $\ker \phi = \{\pm I\}$  and in particular  $\phi$  is injective on a neighbourhood of  $I$ . Because it is a Lie group homomorphism and therefore has constant rank, its differential  $d\phi : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{so}(1, 3)$  is injective. Both Lie algebras have real dimension 6 such that  $d\phi : \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{so}(1, 3)$  gives indeed the sought for isomorphism.

- 4) Show that the symmetric spaces  $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$  and  $\mathbb{H}^n \cong \mathrm{SO}(1, n)^\circ/\mathrm{SO}(n)$  are dual to each other.

*Solution:* Recall that we have seen in the lecture that  $(\mathrm{SO}(n+1), \mathrm{SO}(n), \sigma)$  and  $(\mathrm{SO}(1, n)^\circ, \mathrm{SO}(n), \sigma)$  are Riemannian symmetric pairs where  $\sigma(g) := I_{1,n}gI_{1,n}$  in both cases. Further we have seen that the associated symmetric spaces  $\mathrm{SO}(n+1)/\mathrm{SO}(n)$  and  $\mathrm{SO}(1, n)^\circ/\mathrm{SO}(n)$  are isometric to the  $n$ -sphere  $\mathbb{S}^n$  and (real) hyperbolic  $n$ -space  $\mathbb{H}^n$ . These have  $(\mathfrak{so}(n+1), \zeta)$  and  $(\mathfrak{so}(1, n), \zeta)$  as orthogonal symmetric Lie algebras, respectively, where  $\zeta(X) = d\sigma(X) = I_{1,n}XI_{1,n}$  in both cases.

We have also seen in the lecture that the orthogonal symmetric Lie algebras  $(\mathfrak{so}(p+q), \zeta_{p,q})$  and  $(\mathfrak{so}(p, q), \zeta_{p,q})$  are dual to each other for all  $p, q \geq 1$  where  $\zeta_{p,q}(X) = I_{p,q}XI_{p,q}$  in both cases. Thus for  $p = 1, q = n$  we obtain the assertion.

## References

- [1] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume vol. 34 of *Graduate studies in mathematics*. American Mathematical Society, Providence, R.I, [new ed.] edition, 2001.