The Counting Method of Duke-Rudnick-Sarnak and Eskin-McMullen

Today: *Averaged and asymptotic counting results in a general context.*

Abstract	***Examples(s)***
G linear group	$\mathrm{SL}_2(\mathbb{R})$
Γ lattice	$\mathrm{SL}_2(\mathbb{Z})$
$H < G$ closed, s.t. $\Gamma \cap H < H$ lattice	$\mathbb{U} = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \ : \ t \in \mathbb{R} \right\}$, $\mathrm{SO}(2)$
G/H	$\mathrm{SL}_2(\mathbb{R})/\mathbb{U} \cong \mathbb{R}^2/\{0\}$, $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}$
Count $\lvert \Gamma H \cap B_t \rvert$	$\lvert \mathbb{Z}^2_{\mathrm{prim}} \cap B_t \rvert$, $\lvert \mathrm{SL}_2(\mathbb{Z}) \cdot \mathrm{i} \cap B_t \rvert$

Setup and recall

- Let m_G be a Haar measure on a unimodular group G.
- Let $\Gamma < G$ be a lattice, on which we choose the counting measure as the Haar measure.
- As already known: m_G induces in a natural way a Haar measure m_X on $X = G/\Gamma$ giving total mass $m_X(X) = m_G(F)$ (where F is a Borel fundamental domain for [right action of] Γ).
- Take a closed unimodular subgroup $H < G$ with Haar measure m_H. Then we can define a measure $m_{G/H}$ with following *compatibility properties* [$\backsim$ Fubini]:
  If $f \in L^1_{m_G}(G)$, then
  $$F(gH) := \int_H f(gh) \mathrm{d}m_G(h) \qquad \text{(A)}$$
  exists for almost every $g \in G$, and the measure $m_{G/H}$ satisfies
  $$\int_{G/H} F(gH) \mathrm{d}m_{G/H} = \int_G f \mathrm{d}m_G \qquad \text{(B)} \ .$$

Averaged Counting Result

Dynamical Assumption on X
Let $\Gamma < G$ be a lattice, and assume $H < G$ is a closed subgroup such that $\Gamma \cap H < H$ is a lattice as well. We make the following *equidistribution assumption*:

$$\text{the translated } H\text{-orbits } gH\Gamma \text{ equidistribute in } X = G/\Gamma \text{ as } gH \to \infty \text{ in } G/H. \tag{C}$$

More precisely that means that for $gH \to \infty$ * and for any $\alpha \in C_c(G/\Gamma)$

$$\frac{1}{m_Y(Y)} \int_Y \alpha(gh\Gamma)\ \mathrm{d}m_{H\Gamma}(h\Gamma) \ \longrightarrow \ \frac{1}{m_{G/\Gamma}(G/\Gamma)} \int_{G/\Gamma} \alpha\ \mathrm{d}m_{G/\Gamma}$$

(for $Y = H\Gamma \cong H/\Gamma \cap H \subseteq X = G/\Gamma$).

The above condition (C) implies *equidistribution on average*†:
Let $\{B_t \ : \ t \in \mathbb{R}\} \subset \mathcal{B}(G/H)$ be a collection of subsets of G/H, each with finite Haar measure and such that $m_{G/H}(B_t) \overset{t\to\infty}{\longrightarrow} \infty$. Then (C) implies that for all $\alpha \in C_c(G/\Gamma)$,

$$\frac{1}{m_{G/H}(B_t)} \int_{B_t} \left(\frac{1}{m_Y(Y)} \int_Y \alpha(gh\Gamma)\ \mathrm{d}m_Y(h\Gamma) \right) \mathrm{d}m_{G/H}(gH) \overset{t\to\infty}{\longrightarrow} \frac{1}{m_{G/\Gamma}(G/\Gamma)} \int_{G/\Gamma} \alpha\ \mathrm{d}m_{G/\Gamma}\ . \tag{C'}$$

Proposition 12.7 (Averaged Counting Result).
Let $\Gamma < G$ be a lattice, assume $H < G$ is a closed subgroup such that $\Gamma \cap H < H$ is a lattice as well. Assume that the dynamical assumption (C) holds and let $\{B_t \ : \ t \in \mathbb{R}\} \subset \mathcal{B}(G/H)$ be a collection of subsets of G/H, each with finite Haar measure. Define the *modified orbit-counting function*

$$\begin{aligned} F_t \ : \ X &\longrightarrow \mathbb{R}_{\geq 0} \\ g\Gamma &\longmapsto F_t(g\Gamma) := \frac{1}{m_{G/H}(B_t)} \cdot |g\Gamma H \cap B_t| \end{aligned}$$

If $m_{G/H}(B_t) \overset{t\to\infty}{\longrightarrow} \infty$, then we have the weak*-convergence

$$F_t\ \mathrm{d}m_X \ \overset{t\to\infty}{\longrightarrow} \ \frac{m_{H/\Gamma\cap H}(H/\Gamma \cap H)}{m_X(X)}\ \mathrm{d}m_X\ .$$

Proof. For simplicity of notation we set

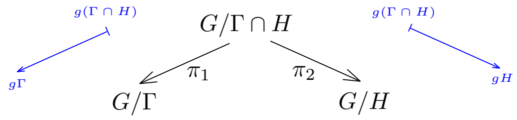
$$\begin{aligned} &Y := H/\Gamma \cap H \\ &m_Y : \text{Haar measure on } Y \text{ induced by } m_H. \end{aligned}$$

We assume the equidistribution assumption (C), and therefore (C') as well.
★ *Wish:* Deduce that

$$\int_X F_t(x)\alpha(x)\ \mathrm{d}m_X \overset{t\to\infty}{\longrightarrow} \frac{m_Y(Y)}{m_X(X)} \int_X \alpha\ \mathrm{d}m_X\ .$$

*I.e. $g_n H \to \infty$, that is to say that $\forall$ compact set K in G/H, $\exists N$ s.t $\forall n > N$, $g_n H \notin K$.
†But the converse direction is not always true.

Main idea of the proof: As Lorraine already did in the her special case, the proof consists of an application of the folding (f.) / unfolding (u.) trick using the spaces



We have

$$A_t^\alpha := \int_X F_t(x)\alpha(x)\ \mathrm{d}m_X$$

$$= \frac{1}{m_{G/H}(B_t)} \int_X |g\Gamma H \cap B_t| \cdot \alpha(g\Gamma)\ \mathrm{d}m_X(g\Gamma)$$

$$= \frac{1}{m_{G/H}(B_t)} \int_{G/\Gamma} \sum_{\gamma \in \Gamma/\Gamma\cap H} \mathbb{1}_{B_t}(g\gamma H) \cdot \alpha(g\Gamma)\ \mathrm{d}m_X(g\Gamma)$$

$$= \frac{1}{m_{G/H}(B_t)} \int_{G/\Gamma} \int_{\Gamma/\Gamma\cap H} \mathbb{1}_{B_t}(g\gamma H) \cdot \alpha(g\Gamma)\ \mathrm{d}m_{\Gamma/\Gamma\cap H}(gH)\ \mathrm{d}m_X(g\Gamma)\ , \qquad \pi_1^{-1}[G/\Gamma] = \Gamma/\Gamma \cap H$$

$$\overset{\text{f.}}{=} \frac{1}{m_{G/H}(B_t)} \int_{G/\Gamma\cap H} \mathbb{1}_{B_t}(gH) \cdot \alpha(g\Gamma)\ \mathrm{d}m_{G/\Gamma\cap H}(g(\Gamma \cap H))$$

$$\overset{\text{u.}}{=} \frac{1}{m_{G/H}(B_t)} \int_{G/H} \mathbb{1}_{B_t}(gH) \int_{H/\Gamma\cap H} \alpha(gh\Gamma)\ \mathrm{d}m_Y(h\Gamma)\ \mathrm{d}m_{G/H}(gH)\ , \qquad \pi_2^{-1}[G/H] = H/\Gamma \cap H$$

$$\overset{\text{def.}}{=} \frac{1}{m_{G/H}(B_t)} \int_{B_t} \int_Y \alpha(gh\Gamma)\ \mathrm{d}m_Y(h\Gamma)\ \mathrm{d}m_{G/H}(Hg)$$

$$\xrightarrow[\text{assumption (C')}]{t\to\infty} \frac{m_Y(Y)}{m_X(X)} \int_X \alpha\ \mathrm{d}m_X\ .$$

$\square$