6.1. Inextendible

Consider the same function $u \in W^{1,\infty}(\Omega)$ as in the solution to problem 5.1. Towards a contradiction, apply the statement of problem 5.5 (a).

6.2. Zero trace and H^1_0

(a) Via partition of unity, reduce the problem to the following model case: Let

$$\begin{aligned} Q &= \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid |x'| < 1 \text{ and } |x_n| < 1\}, \\ Q_+ &= \{x = (x', x_n) \in Q \mid x_n > 0\}. \end{aligned}$$

Given $u \in H^1(Q)$ satisfying $u = 0$ in $Q \setminus Q_+$, prove $\alpha u \in H^1_0(Q_+)$ for any $\alpha \in C^\infty_c(Q)$.

(b) Restrict a suitable function $w \in C^\infty_c(\mathbb{R}^n)$ to the domain from problem 6.1 and exploit continuity of the trace operator (Lemma 8.4.2).

6.3. Ladyženskaja's inequality

Apply the technique from the proof of Sobolev's inequality (Satz 8.5.1) with u^2 in place of u. Conclude using Fubini's theorem and the Cauchy–Schwarz inequality.

6.4. Non-compactness

Exploit translation-invariance of the $W^{1,p}(\mathbb{R}^n)$-norm.

6.5. Compactness

(a) Recall that

- any function $u \in W^{1,p}_0(\Omega)$ can be extended *by zero* to a function $\overline{u} \in W^{1,p}(\mathbb{R}^n)$;
- the space $W^{1,p}(\mathbb{R}^n)$ is reflexive for $1 < p < \infty$;
- the embedding $W^{1,p}(B_R) \hookrightarrow L^p(B_R)$ is compact for any $R > 0$.

Distinguish the cases $p < n$ and $p = n$ and apply Sobolev's embedding (for functions defined on $\mathbb{R}^n$).

(b) How can you adapt the counterexample from problem 6.4?