9.1. Elliptic equations in non-divergence form

Apply Hölder's inequality and the Poincaré inequality to verify that

$$B(u,\varphi) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} \varphi \, dx + \int_{\Omega} cu\varphi \, dx$$

satisfies the conditions for the Lax-Milgram Lemma (Satz 4.3.3) in the space $H^1_0(\Omega)$.

9.2. The reflection Lemma towards boundary regularity

If $\varphi \in C^\infty_c(\mathbb{R}^n)$, then $\psi(x', x_n) := \varphi(x', x_n) - \varphi(x', -x_n)$ satisfies $\psi \in C^\infty \cap H^1_0(\mathbb{R}^n_+)$.

9.3. Horizontal derivatives

Given $u \in H^2(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+)$ and $h \in \mathbb{R} \setminus \{0\}$, let $D_{h,i}u \colon \mathbb{R}^n_+ \to \mathbb{R}$ be given by

$$D_{h,i}u(x) = \frac{u(x + he_i) - u(x)}{h},$$

where $e_i = (0, \dots, 0, 1, 0 \dots, 0, 0) \in \mathbb{R}^n$ has the entry 1 at position $i \in \{1, \dots, n-1\}$. Show $D_{h,i}u \in H^1_0(\mathbb{R}^n_+)$ and prove that there exists a sequence $h_k \xrightarrow{k\to\infty} 0$ such that $D_{h_k,i}u$ converges weakly in $H^1(\mathbb{R}^n_+)$ to some $v \in H^1(\mathbb{R}^n_+)$ as $k \to \infty$. Then show $v \in H^1_0(\mathbb{R}^n_+)$ and argue that $v = \frac{\partial u}{\partial x_i}$.

9.4. Properties of the bilaplacian

(a) Prove $\ker(\Delta^2) = \{0\}$ to conclude injectivity. Apply elliptic regularity twice to conclude surjectivity.

(b) Explain why the boundary terms vanish when integrating by parts.

(c) Exploit parts (a) and (b).

9.5. Weak solutions to the bilaplace equation

(a) By elliptic regularity, $\|u\|_{H^2(\Omega)} \leq C\|\Delta u\|_{L^2}$.

(b) By definition, $H^1_0(\Omega)$ is a closed subspace of $H^1(\Omega)$.

(c) Apply the Riesz representation theorem in the Hilbert space $(H^2(\Omega) \cap H^1_0(\Omega), \langle\cdot,\cdot\rangle)$ to prove existence and uniqueness. Use problem 9.4 (c) to prove regularity.