Part I. Survival kit

2.1. Weak first derivative

Show that $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ given by $u(x) = |x|$ has a weak derivative $u' \in L^1_{\mathrm{loc}}(\mathbb{R})$ given by

$$u'(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

2.2. Weak derivative in $L^p(\Omega)$

(a) Let $\Omega \subset \mathbb{R}^n$ be open, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ a multi-index and $|\alpha| = \sum_{k=1}^n \alpha_k$. Let $u \in L^1_{\mathrm{loc}}(\Omega)$. Given $1 < p \leq \infty$, let $1 \leq q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $D^\alpha u$ exists as weak derivative in $L^p(\Omega)$ if and only if there is a constant C such that

$$\forall \varphi \in C_c^\infty(\Omega): \quad \left| \int_\Omega u\, D^\alpha \varphi \, dx \right| \leq C \|\varphi\|_{L^q(\Omega)}.$$

(b) The assumption $p \neq 1$ in (a) is necessary: Prove that $u = \chi_{]0,1[} \in L^1(\mathbb{R})$ satisfies

$$\forall \varphi \in C_c^\infty(\mathbb{R}): \quad \left| \int_{\mathbb{R}} u\, \varphi' \, dx \right| \leq C \|\varphi\|_{L^\infty(\mathbb{R})}$$

with some $C > 0$, but $u \notin W^{1,1}(\mathbb{R})$, i. e. u does not have a weak derivative in $L^1(\mathbb{R})$.

2.3. Cantor function

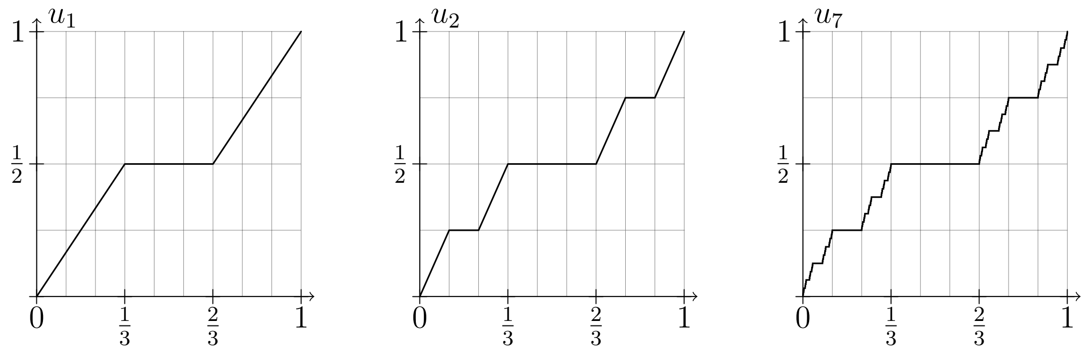
For each $n \in \mathbb{N}$, let $u_n \colon]0,1[\to]0,1[$ be defined iteratively as shown in the figure. The Cantor function $u \colon]0,1[\to]0,1[$ is defined as $u(x) = \lim_{n\to\infty} u_n(x)$.

(a) Show that $u'(x) = 0$ for almost every $x \in]0,1[$.

(b) Construct a sequence of test functions $\varphi_k \in C_c^\infty(]0,1[)$ such that $\varphi_k \to \chi_{]0,1[}$ and

$$-\int_0^1 u\, \varphi_k' \, dx \xrightarrow{k\to\infty} 1.$$

(c) Conclude that the distributional derivative of u does *not* vanish.



Part II. Projects on Green's function for the Laplacian

Definition. Let $n \in \mathbb{N}$. The function $\Phi\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ given by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & (n=2) \\ \frac{1}{n(n-2)|B_1|}|x|^{2-n}, & (n \neq 2) \end{cases}$$

where $|B_1|$ is the volume of the unit ball $B_1 \subset \mathbb{R}^n$ is called *fundamental solution of Laplace's equation.* Let $\Omega \subset \mathbb{R}^n$ be open and bounded with regular boundary. For any fixed $x \in \Omega$ we will show later that there exists a unique solution $\phi^x \in C^\infty(\overline{\Omega})$ to

$$\begin{cases} \Delta\phi^x = 0 & \text{in } \Omega, \\ \phi^x(y) = \Phi(y-x) & \text{for } y \in \partial\Omega. \end{cases}$$

Green's function G for Ω is defined for $x, y \in \Omega$, $x \neq y$ and it is given by

$$G(x,y) = \Phi(y-x) - \phi^x(y).$$

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with regular boundary and Green's function G. Let $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$. Then the function $u\colon \Omega \to \mathbb{R}$ given by*

$$u(x) = -\int_{\partial\Omega} g(x)\,\frac{\partial G}{\partial\nu}(x,y)\,d\sigma(y) + \int_\Omega f(y)\,G(x,y)\,dy,$$

where $\frac{\partial G}{\partial \nu}$ is the derivative of the function $y \mapsto G(x,y)$ in direction of the outer unit normal along $\partial\Omega$, solves the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad\; u = g & \text{on } \partial\Omega. \end{cases}$$

2.4. Symmetry of Green's function ✍

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with regular boundary. Prove that Green's function G for Ω is symmetric in its two variables, i. e.

$$\forall \varphi, \psi \in C_c^\infty(\Omega): \quad \int_\Omega\int_\Omega G(x,y)\varphi(y)\psi(x)\,dx\,dy = \int_\Omega\int_\Omega G(y,x)\varphi(y)\psi(x)\,dx\,dy.$$

2.5. Green's function for the half-space

Show that Green's function for the upper half-space, i. e. the *unbounded* domain $\mathbb{R}^n_+ := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\} \subset \mathbb{R}^n$ exists by computing $G(x,y)$ explicitly.

2.6. Green's function for an interval

(a) Compute Green's function $G(x,y)$ for the one-dimensional domain $]a,b[\subset \mathbb{R}^1$.

(b) Show that $u(x) = \displaystyle\int_a^b G(x,y)f(y)\,dy$ solves $-u'' = f$ in $]a,b[$ with $u(a) = 0 = u(b)$.