Part I. Survival kit

3.1. A closedness property

Let $I :=]a,b[$ for $-\infty \le a < b \le \infty$. Let $u \in L^p(I)$ and let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in the Sobolev space $W^{1,p}(I)$ with $\|u-u_k\|_{L^p(I)} \to 0$ as $k \to \infty$.

(a) If $1 < p \le \infty$, prove $u \in W^{1,p}(I)$.

(b) Is the assumption $p \neq 1$ in part (a) necessary?

3.2. Fundamental solution of Laplace's equation in two dimensions

Given a C^1-function $f\colon \Omega \subseteq \mathbb{R}^2 \to \mathbb{C}$, we define the functions $\frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar z}\colon \Omega \to \mathbb{C}$ by

$$\frac{\partial f}{\partial z} := \frac{1}{2}\Big(\frac{\partial f}{\partial x_1} - i\frac{\partial f}{\partial x_2}\Big), \qquad\qquad \frac{\partial f}{\partial \bar z} := \frac{1}{2}\Big(\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2}\Big).$$

Prove that the function $E\colon \mathbb{R}^2\setminus\{0\} \to \mathbb{C}$ given by $E(x) = \frac{1}{2\pi}\log|x|$ satisfies

(a) $\dfrac{\partial E}{\partial x_j}(x) = \dfrac{x_j}{2\pi|x|^2}$ for $j \in \{1,2\}$ and any $x = (x_1,x_2) \in \mathbb{R}^2\setminus\{0\}$.

(b) $E \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $|\nabla E| \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$.

(c) $\Delta E = \delta_0$ in $\mathcal{D}'(\mathbb{R}^2)$, i. e. $\forall \varphi \in C^\infty_c(\mathbb{R}^2) : \displaystyle\int_{\mathbb{R}^2} E\Delta\varphi\,dx = \varphi(0)$.

(d) $\dfrac{\partial E}{\partial z}(x) = \dfrac{1}{4\pi z}$ for $z := x_1 + ix_2 \in \mathbb{C}\setminus\{0\}$.

(e) For $f \in C^2(\mathbb{R}^2;\mathbb{C})$ notice $\Delta f = 4\dfrac{\partial^2 f}{\partial z\partial\bar z}$. Then prove $\dfrac{\partial}{\partial \bar z}\dfrac{1}{\pi z} = \delta_0$ in $\mathcal{D}'(\mathbb{R}^2)$.

3.3. Linear ODE with constant coefficients

Let $I :=]a,b[$ for $-\infty < a < b < \infty$. Given $f \in C^0(\overline{I})$, consider the equation

$$-u'' + u = f \qquad \text{in } I. \tag{$*$}$$

(a) Show that $(*)$ has a weak solution $u \in H^1_0(I)$ which is unique in $H^1_0(I)$, i. e.

$$\exists!\, u \in H^1_0(I) \quad \forall \varphi \in H^1_0(I) : \quad \int_I u'\varphi'\,dx + \int_I u\varphi\,dx = \int_I f\varphi\,dx.$$

(b) Prove that the weak solution u from (a) is in fact a classical solution $u \in C^2(\overline{I})$.

(c) Given $\alpha, \beta \in \mathbb{R}$ and $g \in C^0(\overline{I})$, deduce that the boundary-value problem

$$\begin{cases} -v'' + v = g & \text{in } I,\\ \quad v(a) = \alpha, \quad v(b) = \beta \end{cases}$$

has a unique classical solution $v \in C^2(\overline{I})$.

3.4. Linear ODE with variable coefficients

Let $I :=]a, b[$ for $-\infty < a < b < \infty$. Let $g \in C^1(\overline{I})$ and $h, f \in C^0(\overline{I})$. Assume that $g(x) \geq \lambda > 0$ and $h(x) \geq 0$ for every $x \in \overline{I}$ and consider the differential equation

$$-(g\,u')' + h\,u = f \quad \text{in } I, \tag{†}$$

(a) Apply the Riesz representation theorem in a suitable Hilbert space to prove that equation (†) has a weak solution $u \in H^1_0(I)$ which is unique in the space $H^1_0(I)$.

(b) Prove that the weak solution u from (a) is in fact a classical solution $u \in C^2(\overline{I})$.

Part II. Projects on Extension operators

3.5. Extension operators of first and second order

Let $1 \leq p \leq \infty$. Recall from the lecture that a continuous linear extension operator $E\colon W^{1,p}(\mathbb{R}_+) \to W^{1,p}(\mathbb{R})$ can be constructed by “even” reflection on the axis $\{x = 0\}$.

Use “odd” reflection, i. e. point reflection in $(0, u(0))$, to construct a linear operator $E\colon W^{2,p}(\mathbb{R}_+) \to W^{2,p}_{\mathrm{loc}}(\mathbb{R})$ satisfying

- $\forall u \in W^{2,p}(\mathbb{R}_+): \quad (Eu)|_{\mathbb{R}_+} = u$ almost everywhere in $\mathbb{R}_+$.
- For every compact subset $K \subset \mathbb{R}$ there is a constant $C > 0$ which is independent of $u \in W^{2,p}(\mathbb{R}_+)$ such that $\|Eu\|_{W^{2,p}(K)} \leq C\|u\|_{W^{2,p}(\mathbb{R}_+)}$.

3.6. Extension operators of any order

(a) Let $k \in \mathbb{N}$. Show that there exist $a_1, \ldots, a_k \in \mathbb{R}$ such that for any polynomial $p\colon \mathbb{R} \to \mathbb{R}$, $p(x) = \sum_{\ell=0}^{k-1} p_\ell x^\ell$ of degree $k-1$ and every $x < 0$, there holds

$$\sum_{j=1}^{k} a_j\, p\Big(\frac{-x}{j}\Big) = p(x).$$

(b) Let $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. Let $a_1, \ldots, a_k \in \mathbb{R}$ as in (a). Prove that the map

$$E\colon u \mapsto Eu, \qquad\qquad (Eu)(x) := \begin{cases} u(x) & \text{for } x > 0, \\ \displaystyle\sum_{j=1}^{k} a_j u\Big(\frac{-x}{j}\Big) & \text{for } x < 0 \end{cases}$$

defines a linear operator $E\colon W^{k,p}(\mathbb{R}_+) \to W^{k,p}(\mathbb{R})$ which allows a constant $C > 0$ such that for every $u \in W^{k,p}(\mathbb{R}_+)$ and any integer $0 \leq \alpha \leq k$

$$\|D^\alpha(Eu)\|_{L^p(\mathbb{R})} \leq C\|D^\alpha u\|_{L^p(\mathbb{R}_+)}.$$