The suggested time for this quiz is 90 minutes. It may be helpful to first work on it without any aid (no books, notes etc...) and then get back, in a second phase, to the more delicate questions with no time constraints and full references at your disposal.

Part I. Multiple choice questions

Find the right answers to the following questions and justify your choice. For each question, there is exactly one correct answer given.

4.1. For what values of p is $u\colon]-1,1[\to \mathbb{R}$ given by $u(x) = |x|$ in $W^{1,p}(]-1,1[)$?

(a) only for $p = 1$.

(b) only for $p = 1$ and $p = 2$.

(c) for all $p \in [1, \infty[$ but not for $p = \infty$.

(d) for all $p \in [1, \infty]$.

(e) None of the above.

4.2. For what values of p is $u\colon \mathbb{R} \to \mathbb{R}$ given by $u(x) = |x|$ in $W^{1,p}(\mathbb{R})$?

(a) only for $p = 1$.

(b) only for $p = 1$ and $p = 2$.

(c) for all $p \in [1, \infty[$ but not for $p = \infty$.

(d) for all $p \in [1, \infty]$.

(e) None of the above.

4.3. Let $n \in \mathbb{N}$ and $B_{\frac{1}{2}} = \{x \in \mathbb{R}^n : |x| < \frac{1}{2}\}$. Given $\alpha \in \mathbb{R}$, let $u_\alpha(x) = \big|\log|x|\big|^\alpha$. What is the set A_n of all $\alpha \in \mathbb{R}$ depending on n such that $u_\alpha \in W^{1,2}(B_{\frac{1}{2}})$?

(a) $A_1 = \{0\}, \quad A_2 =]-\infty, \frac{1}{2}[, \quad A_n = \mathbb{R}$ if $n \geq 3$.

(b) $A_1 = \mathbb{R}, \quad A_2 =]-\infty, \frac{1}{2}[, \quad A_n = \{0\}$ if $n \geq 3$.

(c) $A_n =]-\infty, \frac{n}{2}[$ for any $n \in \mathbb{N}$.

(d) $A_n =]-\infty, 0]$ for any $n \in \mathbb{N}$.

(e) $A_n = \{0\}$ for any $n \in \mathbb{N}$.

4.4. Let $n \in \mathbb{N}$ and $B_{\frac{1}{2}} = \{x \in \mathbb{R}^n : |x| < \frac{1}{2}\}$. Given $\alpha \in \mathbb{R}$, let $u_\alpha(x) = \big|\log|x|\big|^\alpha$. What is the set B_n of all $\alpha \in \mathbb{R}$ depending on n such that $u_\alpha \in W^{1,\infty}(B_{\frac{1}{2}})$?

(a) $B_1 = \{0\}, \quad B_2 =]-\infty, \frac{1}{2}[, \quad B_n = \mathbb{R}$ if $n \geq 3$.

(b) $B_1 = \mathbb{R}, \quad B_2 =]-\infty, \frac{1}{2}[, \quad B_n = \{0\}$ if $n \geq 3$.

(c) $B_n =]-\infty, \frac{n}{2}[$ for any $n \in \mathbb{N}$.

(d) $B_n =]-\infty, 0]$ for any $n \in \mathbb{N}$.

(e) $B_n = \{0\}$ for any $n \in \mathbb{N}$.

4.5. Let $f(x_1, x_2) = x_1 \sin(\frac{1}{x_1}) + x_2 \sin(\frac{1}{x_2})$. Which of the following is true?

(a) $\dfrac{\partial f}{\partial x_1} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ exists as weak derivative.

(b) $\dfrac{\partial f}{\partial x_2} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ exists as weak derivative.

(c) $\dfrac{\partial^2 f}{\partial x_1 \partial x_2} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ exists as weak derivative.

(d) All of the above.

(e) None of the above.

Part II. True or false?

Decide whether the following statements are true or false. If true, formulate a proof. If false, provide a counterexample.

4.6. Let $\mathbb{R}_+ =]0, \infty[\subset \mathbb{R}$. Any $u \in W^{1,2}(\mathbb{R}_+)$ has a bounded representative.

(a) True.

(b) False.

4.7. The weak derivative of any $u \in W^{1,2}(\mathbb{R}_+)$ has a bounded representative.

(a) True.

(b) False.

4.8. Let $I :=]a, b[$ for $-\infty < a < b < \infty$. Then the boundary-value problem

$$\begin{cases} -u'' + u' = f & \text{in } I, \\ \quad u'(a) = 0 = u'(b) \end{cases}$$

has *at least* one weak solution $u \in H^1(I)$ for every $f \in C^0(\overline{I})$.

(a) True.

(b) False.

4.9. Let $I :=]a, b[$ for $-\infty < a < b < \infty$. Then the boundary-value problem

$$\begin{cases} -u'' + u' = f & \text{in } I, \\ \quad u'(a) = 0 = u'(b) \end{cases}$$

has *at most* one weak solution $u \in H^1(I)$ for every $f \in C^0(\overline{I})$.

(a) True.

(b) False.

4.10. Let $n \in \mathbb{N}$ and $B_{\frac{1}{2}} = \{x \in \mathbb{R}^n : |x| < \frac{1}{2}\}$. Given $\alpha \in \mathbb{R}$, let $u_\alpha(x) = \Big|\log|x|\Big|^\alpha$. If α is chosen such that $u_\alpha \in W^{1,2}(B_{\frac{1}{2}})$, then u_α has a representative in $C^0(\overline{B_{\frac{1}{2}}})$.

(a) True.

(b) False.

4.11. Let $n = 1$ and $B_{\frac{1}{2}} = \{x \in \mathbb{R} : |x| < \frac{1}{2}\}$. Given $\alpha \in \mathbb{R}$, let $u_\alpha(x) = \Big|\log|x|\Big|^\alpha$. If α is chosen such that $u_\alpha \in W^{1,2}(B_{\frac{1}{2}})$, then u_α has a representative in $C^1(\overline{B_{\frac{1}{2}}})$.

(a) True.

(b) False.

4.12. Let $I =]-1, 1[$ and let $u, v \colon I \to \mathbb{R}$ be given by $u(x) = |x|$ and $v(x) = (1-x^2)^{\frac{3}{4}}$. Then $uv \in W^{1,3}(I)$.

(a) True.

(b) False.

4.13. The Cantor function on $[0, 1]$ is absolutely continuous.

(a) True.

(b) False.

4.14. $\exists C > 0 \quad \forall u \in H^1(]0,1[) : \quad \int_0^1 |u|^2 \, dx \leq C \int_0^1 |u'|^2 \, dx.$

(a) True.

(b) False.

4.15. Let $0 < a < 1 < b < \infty$ such that $\int_a^b (\log x) \, dx = 0$. Then $\int_a^b |\log x|^2 \, dx \leq \frac{(b-a)^3}{ab}$.

(a) True.

(b) False.