Part I. Survival kit

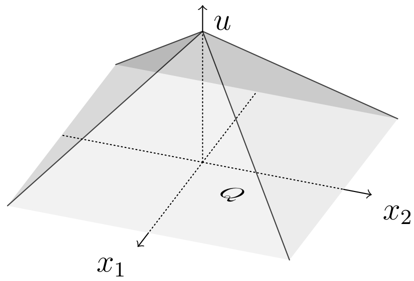
5.1. Lipschitz vs. bounded weak derivative

Find an open set $\Omega \subset \mathbb{R}^2$ and a function $u \in W^{1,\infty}(\Omega)$ which is not Lipschitz continuous.

5.2. A tent for Rudolf L.

Let $Q = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < 1,\ |x_2| < 1\}$. Let $u\colon Q \to \mathbb{R}$ be given by

$$u(x_1,x_2) = \begin{cases} 1 - x_1, & \text{if } x_1 > 0 \text{ and } |x_2| < x_1, \\ 1 + x_1, & \text{if } x_1 < 0 \text{ and } |x_2| < -x_1, \\ 1 - x_2, & \text{if } x_2 > 0 \text{ and } |x_1| < x_2, \\ 1 + x_2, & \text{if } x_2 < 0 \text{ and } |x_1| < -x_2. \end{cases}$$



For which exponents $1 \le p \le \infty$ is $u \in W^{1,p}(Q)$?

5.3. Capacity and Hausdorff measure

Definition (Hausdorff measure). Let $\alpha \ge 0$. For any $\delta > 0$ we define

$$\mathscr{H}^\alpha_\delta(A) := \inf\Big\{\sum_{i=1}^\infty r_i^\alpha : A \subset \bigcup_{i=1}^\infty B_{r_i}(x_i),\ 0 < r_i < \delta,\ x_i \in \mathbb{R}^n\Big\}.$$

The α-dimensional Hausdorff measure of any subset $A \subseteq \mathbb{R}^n$ is defined by

$$\mathscr{H}^\alpha(A) := \lim_{\delta \searrow 0} \mathscr{H}^\alpha_\delta(A)$$

Suppose, $K \subset \mathbb{R}^n$ is a compact subset with $\mathscr{H}^{n-\alpha}(K) = 0$ for some $1 \le \alpha < n$.

(a) For all $1 \le p \le \alpha$, prove that K has vanishing $W^{1,p}$-capacity.

(b) Let $1 \le p \le q \le \infty$ and $\frac{1}{q} + \frac{1}{\alpha} \le 1$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in L^q(\Omega) \cap C^1(\Omega \setminus K)$ with $|\nabla u| \in L^p(\Omega \setminus K)$. Prove that $u \in W^{1,p}(\Omega)$.

Part II. Projects on Traces and Truncations

5.4. Traceless

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with boundary $\partial\Omega$ of class C^1. Let $1 \leq p < \infty$. Prove that there does not exist a continuous linear operator

$$T\colon L^p(\Omega) \to L^p(\partial\Omega)$$

satisfying $Tu = u|_{\partial\Omega}$ for all $u \in C^0(\overline{\Omega})$.

5.5. Traces of weak derivatives

Let $\Omega := \,]0,1[\, \times \,]0,1[\, \subset \mathbb{R}^2$. Given $1 \leq p \leq \infty$, let $u \in W^{1,p}(\Omega)$.

(a) Prove $\big(g\colon x_1 \mapsto u(x_1,x_2)\big) \in W^{1,p}(]0,1[)$ for almost every $x_2 \in \,]0,1[$ with weak derivative

$$g' = \frac{\partial u}{\partial x_1}(\cdot, x_2) \in L^p(]0,1[).$$

(b) Suppose the weak derivatives $\frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_2}$ vanish almost everyhwere in Ω. Using part (a), prove that u has a constant representative.

5.6. Positive and negative part

Let $\Omega \subset \mathbb{R}^n$ be open. Given $1 \leq p < \infty$, let $u \in W^{1,p}(\Omega)$.

(a) Let $u_+(x) = \max\{u(x), 0\}$ and $u_-(x) = -\min\{u(x), 0\}$. Prove $u_+, u_- \in W^{1,p}(\Omega)$ and show that their weak gradients are given by

$$\nabla u_+(x) = \begin{cases} \nabla u(x) & \text{for almost all } x \text{ with } u(x) > 0, \\ 0 & \text{for almost all } x \text{ with } u(x) \leq 0, \end{cases}$$

$$\nabla u_-(x) = \begin{cases} -\nabla u(x) & \text{for almost all } x \text{ with } u(x) < 0, \\ 0 & \text{for almost all } x \text{ with } u(x) \geq 0. \end{cases}$$

(b) Given $u, v \in W^{1,p}(\Omega)$ and $w(x) = \max\{u(x), v(x)\}$ show that $w \in W^{1,p}(\Omega)$.

(c) Prove that $\nabla u(x) = 0$ for almost all $x \in \Omega$ with $u(x) = 0$, which means that if $Z = \{x \in \Omega : u(x) = 0\}$ and $W = \{x \in \Omega : \nabla u(x) = 0 \text{ classically}\}$, then $Z \setminus W$ has Lebesgue measure zero.

(d) Let $\lambda \in \mathbb{R}$. Conclude that $\nabla u(x) = 0$ for almost all $x \in \Omega$ with $u(x) = \lambda$.