The suggested time for this quiz is 90 minutes. It may be helpful to first work on it without any aid (no books, notes etc...) and then get back, in a second phase, to the more delicate questions with no time constraints and full references at your disposal.

Part I. Multiple choice questions

Find the right answers to the following questions and justify your choice. For each question, there is exactly one correct answer given.

8.1. Let $\Omega \subset \mathbb{R}^3$ be open and bounded of class C^1. Into which space does $H^1(\Omega)$ *not* embed continuously?

(a) $C^0(\overline{\Omega})$

(b) $L^4(\Omega)$

(c) $L^6(\Omega)$

(d) $W^{1,1}(\Omega)$

(e) None of the above.

8.2. Let $n \geq 2$ and $p = 2n$. Into which space does $W^{1,p}(\mathbb{R}^n)$ *not* embed continuously?

(a) $L^\infty(\mathbb{R}^n)$

(b) $\mathcal{L}^{p,p}(\mathbb{R}^n)$

(c) $L^n(\mathbb{R}^n)$

(d) $C^{0,\frac{1}{2}}(\mathbb{R}^n)$

(e) None of the above.

8.3. Let $n \geq 2$ and $1 < p < n$. Let $\Omega \subset \mathbb{R}^n$ be bounded of class C^1. Let $u \in W^{1,p}(\Omega)$. Which statement is false?

(a) $u|_{\partial\Omega} \in L^p(\partial\Omega)$ is well-defined.

(b) The embedding $W^{1,p}(\Omega) \hookrightarrow L^{\frac{n}{n-p}}(\Omega)$ is compact.

(c) There exists $C < \infty$ independently of u such that $\|u\|_{W^{1,p}(\Omega)} \leq C\|u\|_{L^{\frac{np}{n-p}}(\Omega)}$.

(d) There exists more than one $v \in W^{1,p}(\mathbb{R}^n)$ with $v|_\Omega = u$.

(e) None of the above.

8.4. Let $n \geq 3$ and let $B_1 \subset \mathbb{R}^n$ be the unit ball. For which $q \geq 1$ is the following inequality true?

$$\exists C < \infty \quad \forall u \in C_c^\infty(B_1): \quad \int_{B_1} |u|^3 \, dx \leq C\Big(\int_{B_1} |\nabla u|^2 \, dx\Big)\Big(\int_{B_1} |u|^q \, dx\Big)^{\frac{2}{n}}$$

(a) any $q \geq 3$

(b) only for $q = 3$

(c) only for $q = \frac{n}{3}$

(d) only for $q = \frac{n}{2}$

(e) None of the above.

8.5. Let $n \in \mathbb{N}$ and let $B_R \subset \mathbb{R}^n$ be the ball of radius $R > 0$ around the origin. Let $1 \leq p < n$ and let $1 \leq q \leq \frac{np}{n-p}$. For which $\beta \in \mathbb{R}$ is the following statement true?

$$\exists C < \infty \quad \forall R > 0 \quad \forall u \in W_0^{1,p}(B_R): \qquad \|u\|_{L^q(B_R)} \leq CR^\beta \|\nabla u\|_{L^p(B_R)}$$

(a) $\beta = 0$

(b) $\beta = 1$

(c) $\beta = \frac{n-p}{q}$

(d) $\beta = \frac{n}{q} - \frac{n}{p} + 1$

(e) None of the above.

Part II. True or false?

Decide whether the following statements are true or false. If true, formulate a proof. If false, provide a counterexample.

8.6. Let $u \in W^{1,1}(\mathbb{R}^n)$. Let $\Omega \subset \mathbb{R}^n$ be any open domain. Then $u|_\Omega \in W^{1,1}(\Omega)$.

(a) True.

(b) False.

8.7. Let $n \in \mathbb{N}$ and $1 \leq p < \infty$. The spaces $W^{1,p}(\mathbb{R}^n)$ and $W_0^{1,p}(\mathbb{R}^n)$ are the same.

(a) True.

(b) False.

8.8. Let $n \geq 2$ and $1 \leq p < n$. The spaces $W^{1,p}(\mathbb{R}^n \setminus \overline{B_1(0)})$ and $W_0^{1,p}(\mathbb{R}^n \setminus \overline{B_1(0)})$ are the same.

(a) True.

(b) False.

8.9. Let $n \geq 2$ and $1 \leq p < n$. The spaces $W^{1,p}(\mathbb{R}^n \setminus \{0\})$ and $W_0^{1,p}(\mathbb{R}^n \setminus \{0\})$ are the same.

(a) True.

(b) False.

8.10. Let $n = 1$ and $p = 1$. The spaces $W^{1,1}(\mathbb{R} \setminus \{0\})$ and $W_0^{1,1}(\mathbb{R} \setminus \{0\})$ are the same.

(a) True.

(b) False.

8.11. Let $\Omega \subset \mathbb{R}^n$ be bounded of class C^1. Then, $C^\infty(\overline{\Omega})$ is dense in $C^{0,\frac{1}{2}}(\overline{\Omega})$.

(a) True.

(b) False.

8.12. Let $B_1 \subset \mathbb{R}^2$ be the unit ball and let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^{1,4}(B_1)$. Then, there exists $v \in C^{0,\frac{1}{4}}(\overline{B_1})$ and $\Lambda \subset \mathbb{N}$ such that $\|v - u_k\|_{C^{0,\frac{1}{4}}(\overline{B_1})} \to 0$ as $\Lambda \ni k \to \infty$.

(a) True.

(b) False.

8.13. There exists functions in $W^{1,3}(\mathbb{R}^2)$ such that all their representatives are nowhere differentiable.

(a) True.

(b) False.

8.14. Every compactly supported function in $W^{1,4}(\mathbb{R}^3)$ is in $L^\infty(\mathbb{R}^3)$.

(a) True.

(b) False.

8.15. Let $B_1 \subset \mathbb{R}^n$ be the unit ball. There exists a sequence $(u_k)_{k\in\mathbb{N}}$ in $C_c^\infty(B_1)$ satisfying $u_k(x) = 1$ for every $x \in B_1$ with $|x| = \frac{1}{2}$ and $\|u_k\|_{W^{1,1}(B_1)} \to 0$ as $k \to \infty$.

(a) True.

(b) False.