2.1. Weak first derivative

Let $\varphi \in C_c^\infty(\mathbb{R})$. Then, since $x\varphi(x)$ vanishes for $x = 0$ and for $x \to \infty$, there holds

$$\begin{aligned}
-\int_{\mathbb{R}} |x|\varphi'(x)\,dx &= -\int_{-\infty}^{0} -x\varphi'(x)\,dx - \int_0^\infty x\varphi'(x)\,dx \\
&= \int_{-\infty}^{0} -1\,\varphi(x)\,dx + \int_0^\infty 1\,\varphi(x)\,dx = \int_{\mathbb{R}} u'(x)\varphi(x)\,dx.
\end{aligned}$$

Since $u' \colon \mathbb{R} \to \mathbb{R}$ given by $u'(x) = 1$ for $x \geq 0$ and $u'(x) = -1$ for $x < 0$ is in $L^1_{\text{loc}}(\mathbb{R})$, the equation above proves that u' is the weak first derivative of $u(x) = |x|$.

2.2. Weak derivative in $L^p(\Omega)$

(a) Let $u \in L^1_{\text{loc}}(\Omega)$. Given $1 < p \leq \infty$, let $1 \leq q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $D^\alpha u$ exists as weak derivative in $L^p(\Omega)$. Let $\varphi \in C_c^\infty(\Omega)$ be arbitrary. Then,

$$\left|\int_\Omega u D^\alpha \varphi\,dx\right| = \left|(-1)^{|\alpha|}\int_\Omega (D^\alpha u)\varphi\,dx\right| \leq \|D^\alpha u\|_{L^p(\Omega)}\|\varphi\|_{L^q(\Omega)}$$

by Hölder's inequality which proves the first claim with constant $C = \|D^\alpha u\|_{L^p(\Omega)}$. Conversely, suppose

$$\forall \varphi \in C_c^\infty(\Omega): \quad \left|\int_\Omega u\, D^\alpha \varphi\,dx\right| \leq C\|\varphi\|_{L^q(\Omega)}.$$

Then, since $C_c^\infty(\Omega)$ is dense in $L^q(\Omega)$ for $q < \infty$, the map

$$f\colon \varphi \mapsto (-1)^{|\alpha|}\int_\Omega u\, D^\alpha\varphi\,dx$$

defines a continuous linear functional $f \in (L^q(\Omega))^*$. Since $(L^q(\Omega))^*$ for $1 \leq q < \infty$ is isometrically isomorphic to $L^p(\Omega)$, there exists $g \in L^p(\Omega)$ such that

$$\forall \varphi \in L^q(\Omega): \quad f(\varphi) = \int_\Omega g\varphi\,dx.$$

By definition of f it follows that $g \in L^p(\Omega)$ is the weak derivative $D^\alpha u$ of u.

(b) Let $u = \chi_{]0,1[}$ and $\varphi \in C_c^\infty(\mathbb{R})$. Then

$$\left|\int_{\mathbb{R}} u\,\varphi'\,dx\right| = \left|\int_0^1 \varphi'\,dx\right| = \big|\varphi(1) - \varphi(0)\big| \leq 2\|\varphi\|_{L^\infty(\mathbb{R})}.$$

The function u restricted to $\mathbb{R} \setminus \{0,1\}$ is differentiable with vanishing derivative. In particular, if u had a weak derivative $u' \in L^1_{\text{loc}}(\mathbb{R})$, then $u' = 0$ almost everywhere. A contradiction arises for test functions $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi(0) \neq \varphi(1)$ via

$$0 = \int_{\mathbb{R}} u'\varphi\,dx = -\int_{\mathbb{R}} u\,\varphi'\,dx = -\int_0^1 \varphi'\,dx = \varphi(0) - \varphi(1).$$

2.3. Cantor function

(a) The set $A_n = \{x \in \,]0,1[\; : \; u_n'(x) \neq 0 \text{ or } u_n'(x) \text{ does not exist classically}\}$ is a union of relatively closed subintervals of equal length. With each iteration $n \rightsquigarrow n+1$ the number of intervals doubles but their length is divided by three. Therefore,

$$\lim_{n\to\infty} |A_n| = \lim_{n\to\infty} (\tfrac{2}{3})^n = 0.$$

By definition of u, we have $\{x \in \,]0,1[\; : \; u'(x) = 0 \text{ exists classically}\} \supset \,]0,1[\, \setminus A_n$ for every $n \in \mathbb{N}$. Thus, $u'(x) = 0$ in a set of full measure, i. e. for almost every $x \in \,]0,1[$.

(b) Given $2 \leq k \in \mathbb{N}$, let $\varphi_k \in C_c^\infty(]0,1[)$ be such that

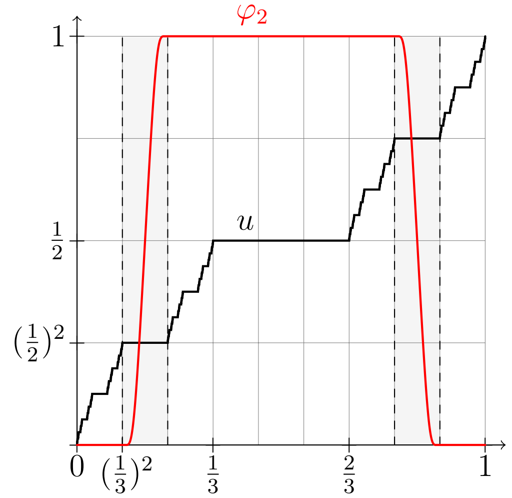
$$\varphi_k(x) = \begin{cases} 0 & \text{for } x \leq (\frac{1}{3})^k, \\ 1 & \text{for } 2(\frac{1}{3})^k \leq x \leq 1 - 2(\frac{1}{3})^k, \\ 0 & \text{for } x \geq 1 - (\frac{1}{3})^k. \end{cases}$$

Then, since

$$u(x) = \begin{cases} (\frac{1}{2})^k & \text{for } (\frac{1}{3})^k < x < 2(\frac{1}{3})^k, \\ 1 - (\frac{1}{2})^k & \text{for } 1 - 2(\frac{1}{3})^k < x < 1 - (\frac{1}{3})^k \end{cases}$$

and since $\varphi'(x)$ vanishes outside this range, there holds

$$\begin{aligned} -\int_0^1 u(x)\varphi_k'(x)\,dx &= -(\tfrac{1}{2})^k \int_{(\frac{1}{3})^k}^{2(\frac{1}{3})^k} \varphi'(x)\,dx - \Big(1 - (\tfrac{1}{2})^k\Big) \int_{1-2(\frac{1}{3})^k}^{1-(\frac{1}{3})^k} \varphi'(x)\,dx \\ &= -(\tfrac{1}{2})^k + \Big(1 - (\tfrac{1}{2})^k\Big) \xrightarrow{k\to\infty} 1. \end{aligned}$$



(c) Suppose the distributional derivative u' of u vanishes. Then $u' = 0$ would be the weak first derivative of u in $L^1(]0,1[)$. However, $\|u'\|_{L^1(]0,1[)} = 0$ is in contradiction to

$$\|u'\|_{L^1(]0,1[)} \geq \lim_{k\to\infty} \int_0^1 u'\varphi_k\,dx = -\lim_{k\to\infty} \int_0^1 u\varphi_k'\,dx = 1.$$

2.4. Symmetry of Green's function

Let G be Green's function for $\Omega \subset \mathbb{R}^n$ and let $\varphi, \psi \in C_c^\infty(\Omega)$ be arbitrary. Consider the functions $u, v \colon \Omega \to \mathbb{R}$ given by

$$u(x) = \int_\Omega G(x,y)\varphi(y)\,dy, \qquad\qquad v(x) = \int_\Omega G(x,y)\psi(y)\,dy.$$

According to the Theorem about Green's function, they satisfy

$$\begin{cases} -\Delta u = \varphi & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \qquad\qquad \begin{cases} -\Delta v = \psi & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore,

$$\begin{aligned}
&\int_\Omega\int_\Omega G(x,y)\varphi(y)\psi(x)\,dx\,dy - \int_\Omega\int_\Omega G(y,x)\varphi(y)\psi(x)\,dx\,dy \\
&= \int_\Omega u(x)\psi(x)\,dx - \int_\Omega v(y)\varphi(y)\,dy \\
&= -\int_\Omega u\Delta v\,dx + \int_\Omega v\Delta u\,dx = -\int_\Omega u\Delta v\,dx + \int_\Omega (\Delta v)u\,dx = 0,
\end{aligned}$$

where we used integration by parts and $v|_{\partial\Omega} = 0 = u|_{\partial\Omega}$ in the last line. Since φ and ψ are arbitrary, symmetry of G follows.

2.5. Green's function for the half-space

Given $x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n_+$, let $\overline{x} = (x_1, \ldots, x_{n-1}, -x_n)$ denote its reflection in the plane $\partial\mathbb{R}^n_+$. Let $\Phi \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be the fundamental solution of Laplace's equation as given on the problem set. Then the function

$$\phi^x(y) := \Phi(y - \overline{x}) = \Phi(y_1 - x_1, \ldots, y_{n-1} - x_{n-1}, y_n + x_n)$$

satisfies

$$\begin{cases} \Delta\phi^x = 0 & \text{in } \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y - x) & \text{for } y \in \partial\mathbb{R}^n_+ \end{cases}$$

because $y - \overline{x} \neq 0$ for every $y \in \mathbb{R}^n_+$ and since by symmetry of Φ

$$\forall y \in \partial\mathbb{R}^n_+ : \quad \Phi(y - x) = \Phi(\overline{y - x}) = \Phi(\overline{y} - \overline{x}) = \Phi(y - \overline{x}) = \phi^x(y).$$

Hence, Green's function for the upper half-space is

$$\begin{aligned}
G(x,y) &= \Phi(y - x) - \phi^x(y) = \Phi(y - x) - \Phi(y - \overline{x}) \\
&= \begin{cases} -\frac{1}{2\pi}\Big(\log|y - x| - \log|y - \overline{x}|\Big), & (n = 2) \\ \frac{1}{n(n-2)|B_1|}\Big(|y - x|^{2-n} - |y - \overline{x}|^{2-n}\Big), & (n \neq 2). \end{cases}
\end{aligned}$$

Remark. Since the domain $\mathbb{R}^n_+$ is unbounded, the representation formula (as given on the problem set) for solutions of the equation $-\Delta u = f$ in $\mathbb{R}^n_+$ with boundary data $u|_{\partial\mathbb{R}^n_+} = g$ has to be checked separately.

2.6. Green's function for an interval

(a) For $n = 1$, the fundamental solution of Laplace's equation is $\Phi\colon \mathbb{R}^1 \to \mathbb{R}$ given by

$\Phi(x) = -\frac{1}{2}|x|$.

Given $x \in \,]a, b[$, it remains to solve the boundary-value problem

$$\begin{cases} (\phi^x)'' = 0 & \text{in }]a, b[, \\ \phi^x(y) = -\frac{1}{2}|x - y| & \text{for } y \in \{a, b\}. \end{cases}$$

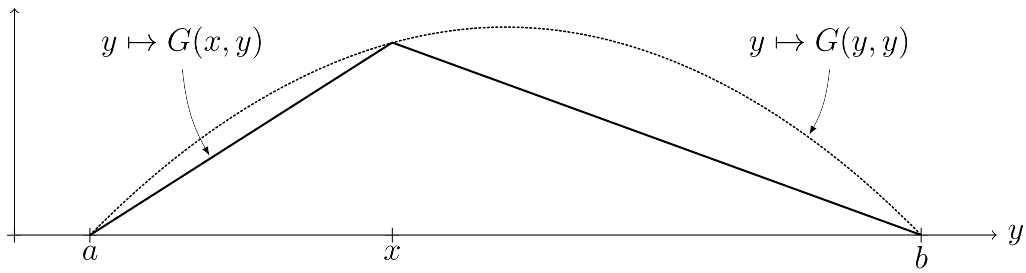
We obtain $\phi^x(y) = c_1 + c_2 y$ with constants $c_1, c_2 \in \mathbb{R}$ determined by the equations

$$\begin{aligned} -\tfrac{1}{2}(x - a) = \phi^x(a) = c_1 + c_2 a \qquad & \Rightarrow \; c_1 = -\tfrac{1}{2}(x - a) - c_2 a, \\ \tfrac{1}{2}(x - b) = \phi^x(b) = c_1 + c_2 b \qquad & \Rightarrow \; c_2(-a + b) = \tfrac{1}{2}(x - a) + \tfrac{1}{2}(x - b). \end{aligned}$$

Hence,

$$\begin{aligned} c_2 &= \frac{(x - a) + (x - b)}{2(b - a)}, \\ c_1 &= -\frac{x - a}{2} - \frac{(x - a)a + (x - b)a}{2(b - a)} = -\frac{(x - a)b + (x - b)a}{2(b - a)}, \end{aligned}$$

$$\begin{aligned} G(x, y) = \Phi(y - x) - c_1 - c_2 y &= -\frac{|y - x|}{2} + \frac{(x - a)(b - y) + (x - b)(a - y)}{2(b - a)} \\ &= \begin{cases} \frac{(x-b)(a-y)}{(b-a)} & \text{if } y \le x, \\ \frac{(x-a)(b-y)}{(b-a)} & \text{if } y > x. \end{cases} \end{aligned}$$



(b) Let $f \in C^0(]a, b[)$ and $u(x) = \displaystyle\int_a^b G(x, y) f(y)\, dy$. Then,

$$\begin{aligned} u'(x) &= \int_a^b \frac{\partial G}{\partial x}(x, y) f(y)\, dy = \int_a^x \frac{(a - y)}{(b - a)} f(y)\, dy + \int_x^b \frac{(b - y)}{(b - a)} f(y)\, dy, \\ u''(x) &= \frac{(a - x)}{(b - a)} f(x) - \frac{(b - x)}{(b - a)} f(x) = \big((a - x) - (b - x)\big) \frac{f(x)}{(b - a)} = -f(x). \end{aligned}$$

Since $G(a, y) = 0 = G(b, y)$ for every $y \in \,]a, b[$, there holds $u(a) = 0 = u(b)$.