Part I. Multiple choice questions

4.1. For what values of p is $u\colon]-1,1[\to \mathbb{R}$ given by $u(x) = |x|$ in $W^{1,p}(]-1,1[)$?

(a) only for $p = 1$.

(b) only for $p = 1$ and $p = 2$.

(c) for all $p \in [1, \infty[$ but not for $p = \infty$.

√ (d) for all $p \in [1, \infty]$.

(e) None of the above.

Being bounded, $u\colon]-1,1[\to \mathbb{R}$ is in $L^p(]-1,1[)$ for any $1 \leq p \leq \infty$. Its weak derivative $u'(x) = \operatorname{sign}(x)$ is also bounded and hence in $L^p(]-1,1[)$ for any $1 \leq p \leq \infty$.

4.2. For what values of p is $u\colon \mathbb{R} \to \mathbb{R}$ given by $u(x) = |x|$ in $W^{1,p}(\mathbb{R})$?

(a) only for $p = 1$.

(b) only for $p = 1$ and $p = 2$.

(c) for all $p \in [1, \infty[$ but not for $p = \infty$.

(d) for all $p \in [1, \infty]$.

√ (e) None of the above.

For any $p \geq 1$, the integral $\int_{\mathbb{R}} |x|^p \, dx$ is infinite.

4.3. Let $n \in \mathbb{N}$ and $B_{\frac{1}{2}} = \{x \in \mathbb{R}^n : |x| < \frac{1}{2}\}$. Given $\alpha \in \mathbb{R}$, let $u_\alpha(x) = \big|\log|x|\big|^\alpha$. What is the set A_n of all $\alpha \in \mathbb{R}$ depending on n such that $u_\alpha \in W^{1,2}(B_{\frac{1}{2}})$?

✓ (a) $A_1 = \{0\}, \quad A_2 =]-\infty, \frac{1}{2}[, \quad A_n = \mathbb{R}$ if $n \geq 3$.

(b) $A_1 = \mathbb{R}, \quad A_2 =]-\infty, \frac{1}{2}[, \quad A_n = \{0\}$ if $n \geq 3$.

(c) $A_n =]-\infty, \frac{n}{2}[$ for any $n \in \mathbb{N}$.

(d) $A_n =]-\infty, 0]$ for any $n \in \mathbb{N}$.

(e) $A_n = \{0\}$ for any $n \in \mathbb{N}$.

In polar coordinates $(r, \theta) \in]0, \frac{1}{2}[\times \mathbb{S}^{n-1}$, we have with $s = \log r$

$$\int_{B_{\frac{1}{2}}} |u_\alpha|^2 \, dx = |\mathbb{S}^{n-1}| \int_0^{\frac{1}{2}} |\log r|^{2\alpha} r^{n-1} dr = |\mathbb{S}^{n-1}| \int_{-\infty}^{\log \frac{1}{2}} |s|^{2\alpha} e^{ns} \, ds,$$

$$\int_{B_{\frac{1}{2}}} |\nabla u_\alpha|^2 \, dx = |\mathbb{S}^{n-1}| \int_0^{\frac{1}{2}} \alpha^2 |\log r|^{2\alpha-2} r^{n-3} dr = |\mathbb{S}^{n-1}| \int_{-\infty}^{\log \frac{1}{2}} \alpha^2 |s|^{2\alpha-2} e^{(n-2)s} \, ds$$

which shows $u_\alpha \in L^2(B_{\frac{1}{2}})$ for any $\alpha \in \mathbb{R}$. If $n \geq 3$, then the second integral also converges for any $\alpha \in \mathbb{R}$. If $n = 2$, then the second integral converges, if $2\alpha - 2 < -1$, that is for $\alpha < \frac{1}{2}$. If $n = 1$, the second integral converges only for $\alpha = 0$.

4.4. Let $n \in \mathbb{N}$ and $B_{\frac{1}{2}} = \{x \in \mathbb{R}^n : |x| < \frac{1}{2}\}$. Given $\alpha \in \mathbb{R}$, let $u_\alpha(x) = \big|\log|x|\big|^\alpha$. What is the set B_n of all $\alpha \in \mathbb{R}$ depending on n such that $u_\alpha \in W^{1,\infty}(B_{\frac{1}{2}})$?

(a) $B_1 = \{0\}, \quad B_2 =]-\infty, \frac{1}{2}[, \quad B_n = \mathbb{R}$ if $n \geq 3$.

(b) $B_1 = \mathbb{R}, \quad B_2 =]-\infty, \frac{1}{2}[, \quad B_n = \{0\}$ if $n \geq 3$.

(c) $B_n =]-\infty, \frac{n}{2}[$ for any $n \in \mathbb{N}$.

(d) $B_n =]-\infty, 0]$ for any $n \in \mathbb{N}$.

✓ (e) $B_n = \{0\}$ for any $n \in \mathbb{N}$.

The function u_α is bounded if and only if $\alpha \leq 0$. Its gradient $|\nabla u_\alpha(x)| = \frac{|\alpha|}{|x|} \big|\log|x|\big|^{\alpha-1}$ is bounded only for $\alpha = 0$.

4.5. Let $f(x_1,x_2)=x_1\sin(\frac{1}{x_1})+x_2\sin(\frac{1}{x_2})$. Which of the following is true?

(a) $\dfrac{\partial f}{\partial x_1}\in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ exists as weak derivative.

(b) $\dfrac{\partial f}{\partial x_2}\in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ exists as weak derivative.

√ (c) $\dfrac{\partial^2 f}{\partial x_1\partial x_2}\in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ exists as weak derivative.

(d) All of the above.

(e) None of the above.

Suppose $\frac{\partial f}{\partial x_1}\in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ exists as weak derivative. Let $\{K_n\}_{n\in\mathbb{N}}$ be a countable collection of compact subsets $K_n=I_n\times J_n\subset\mathbb{R}^2$, where $I_n,J_n:=[-n,n]\subset\mathbb{R}$ are compact intervals, such that $\mathbb{R}^2=\bigcup_{n\in\mathbb{N}}K_n$. Since $\frac{\partial f}{\partial x_1}\in L^1(K_n)$ for any n by assumption, Fubini's theorem implies that $g\colon x_1\mapsto\frac{\partial f}{\partial x_1}(x_1,x_2)$ is in $L^1(I_n)$ for almost every $x_2\in J_n$. Since any compact subset of $\mathbb{R}$ is covered by finitely mand intervals in $\{I_n\}_{n\in\mathbb{N}}$, we conclude that $g\colon x_1\mapsto\frac{\partial f}{\partial x_1}(x_1,x_2)$ is in $L^1_{\mathrm{loc}}(\mathbb{R})$ for almost every $x_2\in\mathbb{R}$.

Let us write $f(x_1,x_2)=v(x_1)+w(x_2)$, where $v(t):=t\sin\frac{1}{t}=:w(t)$. Let $\phi,\psi\in C^\infty_c(\mathbb{R})$ be arbitrary and $\varphi(x_1,x_2)=\phi(x_1)\psi(x_2)$. Since $\varphi\in C^\infty_c(\mathbb{R}^2)$, the definition of weak derivative and Fubini's theorem imply

$$0=\int_{\mathbb{R}^2}\frac{\partial f}{\partial x_1}\varphi+f\frac{\partial\varphi}{\partial x_1}\,dx=\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\frac{\partial f}{\partial x_1}\phi+f\phi'\,dx_1\Big)\psi\,dx_2$$

Since $\psi\in C^\infty_c(\mathbb{R})$ is arbitrary, Satz 3.4.3 (variational Lemma) applies and yields

$$\forall\phi\in C^\infty_c(\mathbb{R})\quad\exists G_\phi\subseteq\mathbb{R}\quad\forall x_2\in G_\phi:\quad 0=\int_{\mathbb{R}}\frac{\partial f}{\partial x_1}\phi+f\phi'\,dx_1$$

and such that the Lebesgue measure of $\mathbb{R}\setminus G_\phi$ vanishes for any ϕ. Let $n\in\mathbb{N}$ and let $\mathcal{P}_n\subset C^\infty_c(]-n,n[)$ be a countable subset, which is dense in the C^1-Topology and $G_n=\bigcap_{\phi\in\mathcal{P}_n}G_\phi$. Then, since $\mathcal{P}_n$ is countable, the Lebesgue measure of $\mathbb{R}\setminus G_n$ still vanishes and we obtain

$$\exists G_n\subseteq\mathbb{R}\quad\forall\phi\in\mathcal{P}_n\quad\forall x_2\in G_n:\quad 0=\int_{\mathbb{R}}\frac{\partial f}{\partial x_1}\phi+f\phi'\,dx_1.\qquad(*)$$

Let $\eta \in C_c^\infty(]-n,n[)$ be arbitrary. By density of $\mathcal{P}_n$ we can choose a sequence $(\phi_k)_{k\in\mathbb{N}}$ of functions $\phi_k \in \mathcal{P}_n$ such that $\|\phi_k - \eta\|_{C^1} \to 0$ as $k \to \infty$ which suffices to pass to the limit in $(*)$. Hence, for all $x_2 \in \bigcap_{n\in\mathbb{N}} G_n$, i.e. for almost all $x_2 \in \mathbb{R}$, there holds

$$0 = \int_{\mathbb{R}} \frac{\partial f}{\partial x_1}\eta + f\eta' \, dx_1 \qquad\qquad \Rightarrow \; -\int_{\mathbb{R}} f\eta' \, dx_1 = \int_{\mathbb{R}} g\eta \, dx_1.$$

for *any* $\eta \in C_c^\infty(\mathbb{R})$ and we conclude that $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ is the weak derivative of $f(\cdot, x_2) \in L^1_{\mathrm{loc}}(\mathbb{R})$ for almost every $x_2 \in \mathbb{R}$. Since the constant function $h\colon x_1 \mapsto w(x_2)$ also has a weak derivative, namely $0 \in L^1_{\mathrm{loc}}(\mathbb{R})$, we conclude by linearity that $v = f(\cdot, x_2) - h \in L^1_{\mathrm{loc}}(\mathbb{R})$ has the weak derivative $g - 0 = g \in L^1_{\mathrm{loc}}(\mathbb{R})$. Hence $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$. This however contradicts the following Lemma.

Lemma. *The continuous function $v\colon [-1,1] \to \mathbb{R}$ given by $v(t) = t\sin\frac{1}{t}$ for $t \neq 0$ and $v(0) = 0$ is not absolutely continuous.*

Proof. For each $k \in \mathbb{N}$ let $t_k = \frac{2}{k\pi} \in \mathbb{R}$. Then, $t_k > t_{k+1}$ for any $k \in \mathbb{N}$ and for any $m, n \in \mathbb{N}$ with $m > n$ there holds

$$\sum_{k=n}^{m-1} (t_k - t_{k+1}) = t_n - t_m < \frac{2}{n\pi}.$$

However,

$$\Big|\tfrac{1}{k}\sin\tfrac{k\pi}{2} - \tfrac{1}{(k+1)}\sin\tfrac{(k+1)\pi}{2}\Big| = \begin{cases} \frac{1}{k} & \text{if } k \text{ is odd,} \\ \frac{1}{k+1} & \text{if } k \text{ is even,} \end{cases}$$

$$\sum_{k=n}^{m-1} \Big| v(t_k) - v(t_{k+1}) \Big| = \frac{2}{\pi}\sum_{k=n}^{m-1} \Big|\tfrac{1}{k}\sin\tfrac{k\pi}{2} - \tfrac{1}{(k+1)}\sin\tfrac{(k+1)\pi}{2}\Big| \geq \frac{2}{\pi}\sum_{k=n}^{m-1}\frac{1}{k+1}.$$

Let $\delta > 0$ be arbitrary. According to the estimates above, we can choose $n, m \in \mathbb{N}$, such that both,

$$\sum_{k=n}^{m-1} (t_k - t_{k+1}) < \delta \qquad \text{and} \qquad \sum_{k=n}^{m-1} \Big| v(t_k) - v(t_{k+1}) \Big| \geq 1,$$

which proves that $v(t) = t\sin\frac{1}{t}$ is not absolutely continuous in $[-1,1]$. □

The argument for non-existence of $\frac{\partial f}{\partial x_2}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ is identical. Thus, f does not have weak first derivatives. However, for any $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} f \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} dx = \int_{\mathbb{R}} v(x_1) \Big(\underbrace{\int_{\mathbb{R}} \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \, dx_2}_{0} \Big) dx_1 + \int_{\mathbb{R}} w(x_2) \Big(\underbrace{\int_{\mathbb{R}} \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \, dx_1}_{0} \Big) dx_2$$

which implies that $\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0 \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ exists as weak derivative.

Part II. True or false?

4.6. Let $\mathbb{R}_+ =]0, \infty[\subset \mathbb{R}$. Any $u \in W^{1,2}(\mathbb{R}_+)$ has a bounded representative.

√ (a) True.

(b) False.

By Sobolev's embedding theorem, $\forall u \in W^{1,2}(\mathbb{R}_+): \ \|u\|_{L^\infty(\mathbb{R}_+)} \leq C\|u\|_{W^{1,2}(\mathbb{R}_+)}$

4.7. The weak derivative of any $u \in W^{1,2}(\mathbb{R}_+)$ has a bounded representative.

(a) True.

√ (b) False.

Functions $u \in W^{1,2}(\mathbb{R}_+)$ have weak derivatives $u \in L^2(\mathbb{R}_+)$ which could be unbounded. For example, $u(x) = x^{\frac{3}{4}}e^{-x}$ with $u'(x) = e^{-x}(\frac{3}{4}x^{-\frac{1}{4}} - x^{\frac{3}{4}})$ satisfies

$$\begin{aligned}
&\|u\|_{L^2(\mathbb{R}_+)} < \infty, \\
&\|u'\|_{L^2(\mathbb{R}_+)} \leq \frac{3}{4}\Big(\int_0^\infty x^{-\frac{1}{2}}e^{-2x}\,dx\Big)^{\frac{1}{2}} + \|u\|_{L^2(\mathbb{R}_+)} < \infty,
\end{aligned}$$

hence $u \in W^{1,2}(\mathbb{R}_+)$ but $u'(x) \to \infty$ for $x \to 0$.

4.8. Let $I :=]a,b[$ for $-\infty < a < b < \infty$. Then the boundary-value problem

$$\begin{cases} -u'' + u' = f & \text{in } I, \\ \quad u'(a) = 0 = u'(b) \end{cases}$$

has *at least* one weak solution $u \in H^1(I)$ for every $f \in C^0(\overline{I})$.

(a) True.

√ (b) False.

Suppose, $u \in H^1(I)$ is a weak solution to the given Neumann problem, i. e.

$$\forall v \in H^1(I): \quad \int_I u'v' + u'v \, dx = \int_I fv \, dx.$$

In particular, u' has the weak derivative $(u' - f) \in L^2(I)$ which implies $u' \in H^1(I)$. But then, $(u')' = (u' - f) \in C^0(\overline{I})$. Hence $u' \in C^1(\overline{I})$ and $u \in C^2(\overline{I})$. So any weak solution is a classical solution. In particular, $w(x) = u'(x)e^{-x}$ satisfies $w(a) = 0$ and

$$\begin{aligned} &-w'(x) = -u''(x)e^{-x} + u'(x)e^{-x} = f(x)e^{-x} \\ \Rightarrow\ & w(x) = w(a) - \int_a^x f(t)e^{-t} \, dt = -\int_a^x f(t)e^{-t} \, dt \\ \Rightarrow\ & u'(x) = -e^x \int_a^x f(t)e^{-t} \, dt, \end{aligned}$$

However, not every $f \in C^0(\overline{I})$ allows u' to satisfy the second boundary condition

$$0 = u'(b) = -e^b \int_a^b f(t)e^{-t} \, dt.$$

4.9. Let $I :=]a,b[$ for $-\infty < a < b < \infty$. Then the boundary-value problem

$$\begin{cases} -u'' + u' = f & \text{in } I, \\ \quad u'(a) = 0 = u'(b) \end{cases}$$

has *at most* one weak solution $u \in H^1(I)$ for every $f \in C^0(\overline{I})$.

(a) True.

√ (b) False.

Let $f = 0$. Then every constant function solves the given boundary-value problem.

4.10. Let $n \in \mathbb{N}$ and $B_{\frac{1}{2}} = \{x \in \mathbb{R}^n : |x| < \frac{1}{2}\}$. Given $\alpha \in \mathbb{R}$, let $u_\alpha(x) = \bigl|\log|x|\bigr|^\alpha$. If α is chosen such that $u_\alpha \in W^{1,2}(B_{\frac{1}{2}})$, then u_α has a representative in $C^0(\overline{B_{\frac{1}{2}}})$.

(a) True.

√ (b) False.

If $n \geq 3$, we may choose any $\alpha > 0$ for which $u_\alpha \in W^{1,2}(B_{\frac{1}{2}})$ is unbounded.

4.11. Let $n = 1$ and $B_{\frac{1}{2}} = \{x \in \mathbb{R} : |x| < \frac{1}{2}\}$. Given $\alpha \in \mathbb{R}$, let $u_\alpha(x) = \bigl|\log|x|\bigr|^\alpha$. If α is chosen such that $u_\alpha \in W^{1,2}(B_{\frac{1}{2}})$, then u_α has a representative in $C^1(\overline{B_{\frac{1}{2}}})$.

√ (a) True.

(b) False.

According to question 4.3, $\alpha = 0$ is the only choice.

4.12. Let $I = \,]-1, 1[$ and let $u, v \colon I \to \mathbb{R}$ be given by $u(x) = |x|$ and $v(x) = (1 - x^2)^{\frac{3}{4}}$. Then $uv \in W^{1,3}(I)$.

√ (a) True.

(b) False.

The function v is differentiable classically in I with derivative $v'(x) = \frac{3}{4}(-2x)(1-x^2)^{-\frac{1}{4}}$. Moreover,

$$\int_{-1}^{1} |v'|^3 \, dx = \frac{27}{8} \int_{-1}^{1} \frac{|x|^3}{(1 - x^2)^{\frac{3}{4}}} \, dx \leq \frac{27}{8} \int_{0}^{1} \frac{2x}{(1 - x^2)^{\frac{3}{4}}} \, dx = \frac{27}{8} \int_{0}^{1} s^{-\frac{3}{4}} \, dx < \infty.$$

Since clearly $v \in L^3(I)$, we obtain $v \in W^{1,3}(I)$. Since also $u \in W^{1,\infty}(I) \subset W^{1,3}(I)$, the statement follows from Corollary 7.3.2.

4.13. The Cantor function on $[0,1]$ is absolutely continuous.

(a) True.

✓ (b) False.

If $u \in C^0([0,1])$ is absolutely continuous, then $u \in W^{1,1}(]0,1[)$. Since the Cantor function has vanishing classical derivative almost everywhere, its weak derivative would be zero which leads to a contradiction as shown on Problem Set 2.

4.14. $\exists C > 0 \quad \forall u \in H^1(]0,1[): \quad \int_0^1 |u|^2\,dx \le C \int_0^1 |u'|^2\,dx.$

(a) True.

✓ (b) False.

The constant functions are in $H^1(]0,1[)$ and do not satisfy the inequality for any C unless they vanish. (Poincaré's inequality requires $u \in H_0^1$ or $\int_I u\,dx = 0$.)

4.15. Let $0 < a < 1 < b < \infty$ such that $\int_a^b (\log x)\,dx = 0$. Then $\int_a^b |\log x|^2\,dx \le \frac{(b-a)^3}{ab}$.

✓ (a) True.

(b) False.

Since $u(x) = \log x$ is smooth in $]a,b[$ satisfying $\int_a^b u(y)\,dy = 0$, there holds

$$
\begin{aligned}
|u(x)| &= \left|\frac{1}{b-a}\int_a^b \Big(u(x)-u(y)\Big)\,dy\right| \\
&\le \frac{1}{b-a}\int_a^b |u(x)-u(y)|\,dy = \frac{1}{b-a}\int_a^b \left|\int_y^x u'(t)\,dt\right| dy \\
&\le \frac{1}{b-a}\int_a^b \int_y^x |u'(t)|\,dt\,dy \\
&\le \frac{1}{b-a}\int_a^b \int_a^b |u'(t)|\,dt\,dy = \int_a^b |u'(t)|\,dt,
\end{aligned}
$$

$$
\begin{aligned}
\int_a^b |u(x)|^2\,dx &\le \int_a^b \left(\int_a^b |u'(t)|\,dt\right)^2 dx \\
&\le \int_a^b \left((b-a)\int_a^b |u'(t)|^2\,dt\right) dx \le (b-a)^2 \int_a^b |u'(t)|^2\,dt,
\end{aligned}
$$

which is the Poincaré inequality. The statement in question is true because

$$
(b-a)^2 \int_a^b |u'(t)|^2\,dt = (b-a)^2 \int_a^b \left|\frac{1}{t}\right|^2 dt = (b-a)^2\left(-\frac{1}{b}+\frac{1}{a}\right) = \frac{(b-a)^3}{ab}.
$$