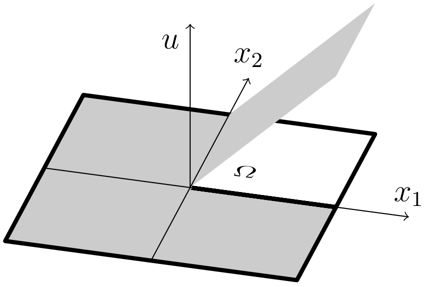
5.1. Lipschitz vs. bounded weak derivative

Let $\Omega = (]-1,1[\times]-1,1[)\setminus([0,1[\times\{0\})$ and let $u\colon\Omega\to\mathbb{R}$ be given by

$$u(x_1,x_2) := \begin{cases} 0 & \text{if } -1 < x_1 \leq 0 \text{ or } x_2 < 0,\\ x_1 & \text{if } x_1 > 0 \text{ and } x_2 > 0.\end{cases}$$



Then, $\Omega\subset\mathbb{R}^2$ is open and u is bounded. For any $\varphi\in C_c^\infty(\Omega)$, we have

$$\begin{aligned}
-\int_\Omega u\frac{\partial\varphi}{\partial x_1}\,dx &= -\int_0^1\Big(\int_0^1 x_1\frac{\partial\varphi}{\partial x_1}\,dx_1\Big)\,dx_2\\
&= \int_0^1\Big(\Big(\int_0^1\varphi\,dx_1\Big) - x_1\varphi(x_1,x_2)\Big|_{x_1=0}^{x_1=1}\Big)\,dx_2 = \int_0^1\int_0^1\varphi\,dx_1\,dx_2,\\
-\int_\Omega u\frac{\partial\varphi}{\partial x_2}\,dx &= -\int_0^1\Big(\int_0^1 x_1\frac{\partial\varphi}{\partial x_2}\,dx_2\Big)\,dx_1 = \int_0^1\Big(0 - x_1\varphi(x_1,x_2)\Big|_{x_2=0}^{x_2=1}\Big)\,dx_1 = 0
\end{aligned}$$

where we used that $(1,x_2),(x_1,1)\in\partial\Omega$ for any $x_1,x_2\in\,]-1,1[$ and $(x_1,0)\in\partial\Omega$ for $x_1>0$ which implies that φ vanishes at these points. Hence, the weak derivatives $\frac{\partial u}{\partial x_1} = \chi_{]0,1[^2}\in L^\infty(\Omega)$ and $\frac{\partial u}{\partial x_2} = 0\in L^\infty(\Omega)$ exist and $u\in W^{1,\infty}(\Omega)$. However, since

$$\frac{|u(\frac12,-\frac1k)-u(\frac12,\frac1k)|}{|(\frac12,-\frac1k)-(\frac12,\frac1k)|} = \frac{\frac12}{\frac2k} = \frac{k}{4}$$

is well-defined for any $k>1$ and unbounded for $k\to\infty$, we conclude that u is not Lipschitz continuous.

5.2. A tent for Rudolf Lipschitz

The function $u\colon Q\to\mathbb{R}$ is given by $u(x_1,x_2) = 1-\max\{|x_1|,|x_2|\}$ and it is bounded in Q. Let $x=(x_1,x_2), y=(y_1,y_2)\in Q$ be arbitrary; w.l.o.g. $u(y)>u(x)$. Then

$$\begin{aligned}
u(y)-u(x) &= \max\{|x_1|,|x_2|\} - \max\{|y_1|,|y_2|\}\\
&\leq \begin{cases} |x_1|-|y_1| \leq |x_1-y_1| & \text{if } |x_1|\geq|x_2|,\\ |x_2|-|y_2|\leq|x_2-y_2| & \text{if } |x_1|<|x_2|\end{cases}\\
&\leq |x-y|
\end{aligned}$$

which implies that u is Lipschitz continuous. Hence $u\in W^{1,\infty}(Q)$. Since Q is bounded, $u\in W^{1,\infty}(Q)\subset W^{1,p}(Q)$ for any $1\leq p\leq\infty$.

5.3. Capacity and Hausdorff measure

Let $K \subset \mathbb{R}^n$ be compact with Hausdorff measure $\mathscr{H}^{n-\alpha}(K) = 0$ for some $1 \leq \alpha < n$.

(a) Let $1 \leq p \leq \alpha$. Let $\varepsilon > 0$. By definition, there exists a collection of points $x_i \in \mathbb{R}^n$ and radii $0 < r_i < 1$ such that

$$K \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i), \qquad \sum_{i=1}^{\infty} r_i^{n-\alpha} < \varepsilon.$$

Since K is compact, there exists $N \in \mathbb{N}$ such that

$$K \subset \bigcup_{i=1}^{N} B_{r_i}(x_i), \qquad \sum_{i=1}^{N} r_i^{n-\alpha} < \varepsilon.$$

For every $i \in \{1, \ldots, N\}$ there exists a function $\psi_i \in C_c^\infty(\mathbb{R}^n)$ satisfying

$$\psi_i = 0 \quad \text{in } \mathbb{R}^n \setminus B_{3r_i}(x_i), \qquad \psi_i = 1 \quad \text{in } B_{2r_i}(x_i), \qquad |\nabla \psi_i| \leq \frac{2}{r_i}.$$

Let $\phi(x) := \max\{\psi_1(x), \ldots, \psi_N(x)\}$. Then, $\phi \in W^{1,p}$ as shown in Problem 5.6 (b). Moreover, there exists a constant C depending only on n and p such that

$$\int_{\mathbb{R}^n} |\nabla \phi|^p \, dx \leq \sum_{i=1}^{N} \int_{B_{3r_i}(x_i)} |\nabla \psi_i|^p \, dx \leq \sum_{i=1}^{N} C r_i^{n-p} \leq \sum_{i=1}^{N} C r_i^{n-\alpha} < C\varepsilon,$$

where we used $r_i^{-p} \leq r_i^{-\alpha}$ for $p \leq \alpha$ and $r_i < 1$. Let $r_0 := \min\{r_1, \ldots, r_N\}$ and let $0 \leq \rho \in C_c^\infty(B_{r_0}(0))$ with $\int_{\mathbb{R}^n} \rho \, dx = 1$. Then the mollification $\varphi := \rho * \phi \in C_c^\infty(\mathbb{R}^n)$ has the property that for any $i \in \{1, \ldots, N\}$ and all $x \in B_{r_i}(x_i)$

$$\varphi(x) = \int_{\mathbb{R}^n} \rho(y)\phi(x-y) \, dy = \int_{B_{r_0}(0)} \rho(y)\phi(x-y) \, dy = \int_{B_{r_0}(0)} \rho(y) \, dy = 1,$$

as $|(x-y) - x_i| \leq |x - x_i| + |y| < r_i + r_0 < 2r_i$ for all $x \in B_{r_i}(x_i)$ and all $y \in B_{r_0}(0)$. Hence, $\varphi = 1$ in $\bigcup_{i=1}^N B_{r_i}(x_i) \supset K$. Furthermore,

$$\|\nabla \varphi\|_{L^p(\mathbb{R}^n)} = \|\rho * \nabla \phi\|_{L^p(\mathbb{R}^n)} \leq \|\rho\|_{L^1(\mathbb{R}^n)} \|\nabla \phi\|_{L^p(\mathbb{R}^n)} = \|\nabla \phi\|_{L^p(\mathbb{R}^n)} \leq (C\varepsilon)^{\frac{1}{p}}.$$

For every $k \in \mathbb{N}$, let φ_k be the function φ constructed above for the choice $\varepsilon = \frac{1}{k} > 0$. Then $\|\nabla \varphi_k\|_{L^p(\mathbb{R}^n)} \to 0$ as $k \to \infty$. By construction, $\varphi_k(x) \to 0$ for every $x \in \mathbb{R}^n \setminus K$. In particular, $\varphi_k(x) \to 0$ for almost every $x \in \mathbb{R}^n$ because $\mathscr{H}^{n-\alpha}(K) = 0$ implies that K has vanishing Lebesgue measure. Since $\varphi_k = 1$ in a neighbourhood of K, we have shown that K has vanishing $W^{1,p}$-capacity.

(b) Let $1 \leq p \leq q \leq \infty$ and $\frac{1}{q} + \frac{1}{\alpha} \leq 1$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in L^q(\Omega) \cap C^1(\Omega \setminus K)$ with $|\nabla u| \in L^p(\Omega \setminus K)$. Let $1 \leq s \leq \infty$ such that $\frac{1}{q} + \frac{1}{s} = 1$. Then, $s \leq \alpha$ which by (a) implies $\operatorname{cap}_{W^{1,s}}(K) = 0$. By Satz 8.1.1, $u \in W^{1,p}(\Omega)$ as claimed.

5.4. Traceless

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with boundary of class C^1 and $1 \leq p < \infty$. Suppose, $T\colon L^p(\Omega) \to L^p(\partial\Omega)$ is a continuous linear operator satisfying $Tu = u|_{\partial\Omega}$ for all $u \in C^0(\overline{\Omega}) \subset L^p(\Omega)$. Given $k \in \mathbb{N}$, let $u_k\colon \overline{\Omega} \to \mathbb{R}$ be given by

$$u_k(x) = \begin{cases} 1 - k\,\mathrm{dist}(x, \partial\Omega), & \text{if } \mathrm{dist}(x, \partial\Omega) \leq \frac{1}{k}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $u_k \in C^0(\overline{\Omega})$ and $u_k|_{\partial\Omega} \equiv 1$. Moreover, $u_k(x) \to 0$ as $k \to \infty$ for almost every $x \in \Omega$ and $|u_k| \leq 1 \in L^p(\Omega)$ for every $k \in \mathbb{N}$. Since $1 \leq p < \infty$, the dominated convergence theorem implies $\|u_k\|_{L^p(\Omega)} \to 0$ as $k \to \infty$. But $\|Tu_k\|_{L^p(\partial\Omega)} = \|1\|_{L^p(\partial\Omega)}$ does not converge to zero which contradicts continuity of T.

5.5. Traces of weak derivatives

Let $\Omega := \,]0,1[\, \times \,]0,1[\, \subset \mathbb{R}^2$. Given $1 \leq p \leq \infty$, let $u \in W^{1,p}(\Omega)$.

(a) In the case $1 \leq p < \infty$ we have $|u|^p \in L^1(\Omega)$ by assumption and Fubini's theorem implies that the map $x_1 \mapsto |u(x_1, x_2)|^p$ is in $L^1(]0,1[)$ for almost every $x_2 \in \,]0,1[$. Hence, $g := u(\cdot, x_2) \in L^p(]0,1[)$ and analogously, $f := \frac{\partial u}{\partial x_1}(\cdot, x_2) \in L^p(]0,1[)$ for almost every $x_2 \in \,]0,1[$.

In the case $p = \infty$ we know that $u \in W^{1,\infty}(\Omega)$ has a (globally) Lipschitz continuous representative because Ω is convex. In particular, $g = u(\cdot, x_2)$ has a Lipschitz continuous representative for almost every $x_2 \in \,]0,1[$. Hence, $g \in W^{1,\infty}(]0,1[)$ for almost every $x_2 \in \,]0,1[$.

It remains to prove that f is actually the weak derivative of g for almost all $x_2 \in \,]0,1[$. Let $\phi, \psi \in C_c^\infty(]0,1[)$ and let $\varphi(x_1, x_2) = \phi(x_1)\psi(x_2)$. Then, since $\varphi \in C_c^\infty(\Omega)$,

$$0 = \int_\Omega \frac{\partial u}{\partial x_1}\varphi + u\frac{\partial \varphi}{\partial x_1}\,dx = \int_0^1 \Bigl(\int_0^1 \frac{\partial u}{\partial x_1}\phi + u\phi'\,dx_1\Bigr)\psi\,dx_2.$$

Since $\psi \in C_c^\infty(]0,1[)$ is arbitrary, Satz 3.4.3 (variational Lemma) applies and yields

$$\forall \phi \in C_c^\infty(]0,1[) \quad \exists G_\phi \subseteq \,]0,1[\quad \forall x_2 \in G_\phi: \quad 0 = \int_0^1 \frac{\partial u}{\partial x_1}\phi + u\phi'\,dx_1$$

and such that the Lebesgue measure of $]0,1[\setminus G_\phi$ vanishes for any ϕ. Let $\mathcal{P} \subset C_c^\infty(]0,1[)$ be a countable subset, which is dense in the C^1-Topology and $G = \bigcap_{\phi\in\mathcal{P}} G_\phi$. Then, since $\mathcal{P}$ is countable, the Lebesgue measure of $]0,1[\setminus G$ still vanishes and we obtain

$$\exists G \subseteq \,]0,1[\quad \forall \phi \in \mathcal{P} \quad \forall x_2 \in G: \quad 0 = \int_0^1 \frac{\partial u}{\partial x_1}\phi + u\phi'\,dx_1. \tag{$*$}$$

Let $\eta \in C_c^\infty(]0,1[)$ be arbitrary. By density of $\mathcal{P}$ we can choose a sequence of functions $\phi_k \in \mathcal{P}$ such that $\|\phi_k - \eta\|_{C^1} \to 0$ as $k \to \infty$ which suffices to pass to the limit in $(*)$. Hence, for all $x_2 \in G$, i.e. for almost all $x_2 \in]0,1[$, there holds

$$\forall \eta \in C_c^\infty(]0,1[): \quad 0 = \int_0^1 \frac{\partial u}{\partial x_1}\eta + u\eta' \, dx_1 \qquad \Rightarrow \ -\int_0^1 g\eta' \, dx_1 = \int_0^1 f\eta \, dx_1$$

which implies that f is the weak derivative of g for almost all $x_2 \in]0,1[$ as claimed.

(b) Suppose the weak derivatives $\frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_2}$ vanish almost everywhere in Ω. By part (a), $g := u(\cdot, x_2) \in W^{1,p}(]0,1[)$ with weak derivative $g' = 0$ for almost every $x_2 \in]0,1[$. Hence, g has a constant representative, i.e. $u(\cdot, x_2) = C(x_2)$ for almost all $x_2 \in]0,1[$. Analogously, $u(x_1, \cdot) = \tilde{C}(x_1)$ for almost all $x_1 \in]0,1[$. Since $C(x_2) = \int_0^1 u(x_1,x_2)\,dx_1$ and $\tilde{C}(x_1) = \int_0^1 u(x_1,x_2)\,dx_2$ are measurable, by Fubini's theorem we have $C(x_2) = u(x_1,x_2) = \tilde{C}(x_1)$ for almost every point (x_1,x_2). Hence, for almost every fixed x_1, we have $C(x_2) = \tilde{C}(x_1)$ for almost every x_2. So C is constant almost everywhere and thus u has a constant representative.

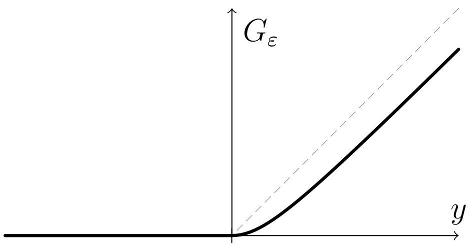
5.6. Positive and negative part

Let $\Omega \subset \mathbb{R}^n$ be open. Given $1 \leq p < \infty$, let $u \in W^{1,p}(\Omega)$.

(a) In order to prove $u_+ \in W^{1,p}(\Omega)$, we consider the function $G_\varepsilon \in C^1(\mathbb{R})$ and its derivative G_ε' given by

$$G_\varepsilon(y) = \begin{cases} \sqrt{y^2+\varepsilon^2} - \varepsilon & \text{for } y \geq 0, \\ 0 & \text{for } y < 0, \end{cases}$$

$$G_\varepsilon'(y) = \begin{cases} \frac{y}{\sqrt{y^2+\varepsilon^2}} & \text{for } y \geq 0, \\ 0 & \text{for } y < 0 \end{cases}$$



for some $\varepsilon > 0$. Then, $G_\varepsilon(0) = 0$ and $|G_\varepsilon'| < 1$. By the chain rule, $G_\varepsilon \circ u \in W^{1,p}(\Omega)$ with weak gradient $\nabla(G_\varepsilon \circ u) = (G_\varepsilon' \circ u)\nabla u \in L^p(\Omega)$. Since $|G_\varepsilon \circ u| \leq |u| \in L^p(\Omega)$ and since $(G_\varepsilon \circ u)(x) \to u_+(x)$ as $\varepsilon \to 0$ pointwise almost everywhere, Lebesgue's dominated convergence theorem implies that $\|u_+ - (G_\varepsilon \circ u)\|_{L^p(\Omega)} \to 0$ as $\varepsilon \to 0$. Similarly, $|\nabla(G_\varepsilon \circ u)| = |G_\varepsilon' \circ u||\nabla u| \leq |\nabla u| \in L^p(\Omega)$. If $u(x) > 0$, then $G_\varepsilon'(u(x)) \to 1$ as $\varepsilon \to 0$. Otherwise, $G_\varepsilon'(u(x)) = 0$. Therefore, we have pointwise convergence

$$\nabla(G_\varepsilon \circ u)(x) \xrightarrow{\varepsilon \to 0} g(x) := \begin{cases} \nabla u(x) & \text{for almost all } x \text{ with } u(x) > 0, \\ 0 & \text{for almost all } x \text{ with } u(x) \leq 0 \end{cases}$$

and after application of the dominated convergence theorem, $\|g - \nabla(G_\varepsilon \circ u)\|_{L^p(\Omega)} \to 0$ as $\varepsilon \to 0$. Since the space $W^{1,p}(\Omega)$ is complete, and since $(G_\varepsilon \circ u)$ converges (for a sequence $\varepsilon \to 0$) in $W^{1,p}(\Omega)$, we conclude $u_+ \in W^{1,p}(\Omega)$ with weak gradient $\nabla u_+ = g$. The proof of $u_- \in W^{1,p}(\Omega)$ is identical after replacing $G_\varepsilon(y)$ with $G_\varepsilon(-y)$.

(b) Let $u, v \in W^{1,p}(\Omega)$. Then, $(u-v)_+ \in W^{1,p}(\Omega)$ by part (a). Since

$$w(x) := \max\{u(x), v(x)\} = \max\{u(x) - v(x), 0\} + v(x),$$

we have $w = (u-v)_+ + v \in W^{1,p}(\Omega)$.

(c) Any $u \in W^{1,p}(\Omega)$ satisfies $u = u_+ - u_-$ with weak gradient $\nabla u = \nabla u_+ - \nabla u_-$. Part (a) implies in particular, that $\nabla u_+(x) = 0$ and $\nabla u_+(x) = 0$ for almost all $x \in \Omega$ with $u(x) = 0$. Consequently, $\nabla u(x) = 0$ for almost all $x \in \Omega$ with $u(x) = 0$.

(d) Given $\lambda \in \mathbb{R}$ we define $u_\lambda(x) = u(x) - \lambda$. However, unless Ω is bounded, we only have $u_\lambda \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Let $r \geq 1$. Then, $u_\lambda \in W^{1,p}(\Omega \cap B_r)$. By part (c), $\nabla u(x) = \nabla u_\lambda(x) = 0$ for almost all $x \in \Omega \cap B_r$ with $u_\lambda(x) = 0$. Since a countable union of sets of measure zero still has measure zero and since $\Omega = \bigcup_{r \in \mathbb{N}} (\Omega \cap B_r)$ we conclude that $\nabla u(x) = 0$ for almost all $x \in \Omega$ with $u(x) = \lambda$.