

7.1. Completeness of Campanato spaces

Let $(u_k)_{k \in \mathbb{N}}$ be a Cauchy sequence in the Campanato space $(\mathcal{L}^{p,\lambda}(\Omega), \|\cdot\|_{\mathcal{L}^{p,\lambda}(\Omega)})$. In particular, $(u_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in the space $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ which is complete. Hence there exists $v \in L^p(\Omega)$ such that

$$\lim_{k \rightarrow \infty} \|u_k - v\|_{L^p(\Omega)} = 0.$$

It remains to prove $v \in \mathcal{L}^{p,\lambda}(\Omega)$ and $\lim_{k \rightarrow \infty} [u_k - v]_{\mathcal{L}^{p,\lambda}} = 0$. Let $x_0 \in \Omega$ and $0 < r < r_0$. Since by Hölder's inequality

$$|(u_m)_{x_0,r} - v_{x_0,r}|^p = \left| \int_{\Omega \cap B_r(x_0)} u_m - v \, dx \right|^p \leq \int_{\Omega \cap B_r(x_0)} |u_m - v|^p \, dx \xrightarrow{m \rightarrow \infty} 0,$$

we conclude that $(u_m - (u_m)_{x_0,r})$ converges to $(v - v_{x_0,r})$ in $L^p(\Omega \cap B_r(x_0))$ as $m \rightarrow \infty$. In particular,

$$\begin{aligned} r^{-\frac{\lambda}{p}} \|v - v_{x_0,r}\|_{L^p(\Omega \cap B_r(x_0))} &= \lim_{m \rightarrow \infty} r^{-\frac{\lambda}{p}} \|u_m - (u_m)_{x_0,r}\|_{L^p(\Omega \cap B_r(x_0))} \\ &\leq \limsup_{m \rightarrow \infty} [u_m]_{\mathcal{L}^{p,\lambda}}. \end{aligned} \quad (1)$$

Since $(u_m)_{m \in \mathbb{N}}$ being Cauchy in $\mathcal{L}^{p,\lambda}(\Omega)$ implies that (1) is finite, and since $x_0 \in \Omega$ and $0 < r < r_0$ are arbitrary, $[v]_{\mathcal{L}^{p,\lambda}} < \infty$ follows. Hence, $v \in \mathcal{L}^{p,\lambda}(\Omega)$.

Let $\varepsilon > 0$. By assumption, there exists $N_\varepsilon \in \mathbb{N}$ such that $[u_n - u_m]_{\mathcal{L}^{p,\lambda}} < \varepsilon$ for all $n, m \geq N_\varepsilon$ which implies that for every $x_0 \in \Omega$ and for all $0 < r < r_0$ and $n, m \geq N_\varepsilon$

$$r^{-\frac{\lambda}{p}} \|u_n - (u_n)_{x_0,r} - u_m + (u_m)_{x_0,r}\|_{L^p(\Omega \cap B_r(x_0))} < \varepsilon. \quad (2)$$

As in (1), we may pass to the limit $m \rightarrow \infty$ in (2) and obtain

$$r^{-\frac{\lambda}{p}} \|u_n - (u_n)_{x_0,r} - v + v_{x_0,r}\|_{L^p(\Omega \cap B_r(x_0))} < \varepsilon \quad (3)$$

for every $n \geq N_\varepsilon$. Since $x_0 \in \Omega$ and $0 < r < r_0$ are arbitrary, we conclude $[u_n - v]_{\mathcal{L}^{p,\lambda}} < \varepsilon$ for every $n \geq N_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\|u_n - v\|_{\mathcal{L}^{p,\lambda}} \rightarrow 0$ as $n \rightarrow \infty$ follows.

7.2. Vanishing weak gradient

Let $1 \leq p \leq \infty$. Let $u \in W^{1,p}(\Omega)$ satisfy $\nabla u = 0 \in L^p(\Omega)$. Since $\Omega \subset \mathbb{R}^n$ is connected and bounded of class C^1 , there exists $\varepsilon_0 > 0$ such that $\Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\}$ is connected for every $0 < \varepsilon < \varepsilon_0$. Fix $0 < \varepsilon_1 < \varepsilon_0$ and let $\rho_\varepsilon \in C_c^\infty(B_\varepsilon(0))$ be a standard mollifier for $0 < \varepsilon < \varepsilon_1$. Then the mollification $u_\varepsilon = u * \rho_\varepsilon$ is well-defined and smooth in Ω_{ε_1} and satisfies $\nabla u_\varepsilon = 0$ in Ω_{ε_1} classically, i. e. u_ε is constant in Ω_{ε_1} .

Moreover, $\|u - u_\varepsilon\|_{L^1(\Omega_{\varepsilon_1})} \rightarrow 0$ as $\varepsilon \rightarrow 0$ implies that the constants $u_\varepsilon|_{\Omega_{\varepsilon_1}}$ converge:

$$\left| u_\varepsilon - \int_{\Omega_{\varepsilon_1}} u \, dx \right| = \left| \int_{\Omega_{\varepsilon_1}} u_\varepsilon - u \, dx \right| \leq \int_{\Omega_{\varepsilon_1}} |u_\varepsilon - u| \, dx \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Since $\|u - u_\varepsilon\|_{L^1(\Omega_{\varepsilon_1})} \rightarrow 0$ implies pointwise convergence almost everywhere on a subsequence, we obtain $u(x_0) = \int_{\Omega_{\varepsilon_1}} u \, dx$ by uniqueness of limits for almost every $x_0 \in \Omega_{\varepsilon_1}$. Letting $\varepsilon_1 \rightarrow 0$ completes the proof.

Remark. The statement generalises to arbitrary connected, open sets $\Omega \subset \mathbb{R}^n$.

7.3. Hölder continuity of functions in $W^{2,n}$

Let $0 < \alpha < 1$ be arbitrary. Let $u \in W^{2,n}(\mathbb{R}^n)$. Then u and $\partial_j u$ are in $W^{1,n}(\mathbb{R}^n)$ for any $j \in \{1, \dots, n\}$. For any $n \leq p < \infty$, especially for $p = \frac{n}{1-\alpha}$, we have the embedding $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$. Hence, u and $\partial_j u$ are in $L^{\frac{n}{1-\alpha}}(\mathbb{R}^n)$ for any $j \in \{1, \dots, n\}$, which shows $u \in W^{1, \frac{n}{1-\alpha}}(\mathbb{R}^n)$. We conclude via the embedding $W^{1, \frac{n}{1-\alpha}}(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$.

7.4. Uniform bounds on functions in $W^{n,1}$

Let $u \in C_c^\infty(\mathbb{R}^n)$ and let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ be arbitrary. Then,

$$\begin{aligned} u(x_1, \dots, x_n) &= \int_{-\infty}^{x_1} \frac{\partial u}{\partial x_1}(s_1, x_2, \dots, x_n) \, ds_1 \\ &= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \frac{\partial^2 u}{\partial x_2 \partial x_1}(s_1, s_2, x_2, \dots, x_n) \, ds_2 \, ds_1 \\ &= \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} \frac{\partial^n u}{\partial x_n \dots \partial x_1}(s_1, \dots, s_n) \, ds_n \dots ds_1, \\ \Rightarrow |u(x)| &\leq \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \left| \frac{\partial^n u}{\partial x_n \dots \partial x_1}(s_1, \dots, s_n) \right| \, ds_n \dots ds_1 \leq \|u\|_{W^{n,1}(\mathbb{R}^n)}. \end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary,

$$\forall u \in C_c^\infty(\mathbb{R}^n) : \quad \|u\|_{L^\infty(\mathbb{R}^n)} \leq \|u\|_{W^{n,1}(\mathbb{R}^n)} \tag{4}$$

follows. The inequality (4) remains true for arbitrary $u \in W^{n,1}(\mathbb{R}^n)$ by density of $C_c^\infty(\mathbb{R}^n)$ in $W^{n,1}(\mathbb{R}^n)$. Indeed, given $u \in W^{n,1}(\mathbb{R}^n)$, let $(u_k)_{k \in \mathbb{N}}$ be a sequence in $C_c^\infty(\mathbb{R}^n)$ such that $\|u_k - u\|_{W^{n,1}(\mathbb{R}^n)} \rightarrow 0$ as $k \rightarrow \infty$. Since inequality (4) implies $\|u_k - u_m\|_{L^\infty(\mathbb{R}^n)} \leq \|u_k - u_m\|_{W^{n,1}(\mathbb{R}^n)}$ the sequence $(u_k)_{k \in \mathbb{N}}$ is Cauchy in $L^\infty(\mathbb{R}^n)$ and hence convergent to some v in $L^\infty(\mathbb{R}^n)$. In particular, $u_k(x) \rightarrow v(x)$ converges pointwise for almost every $x \in \mathbb{R}^n$ as $k \rightarrow \infty$. Moreover, since $\|u_k - u\|_{L^n(\mathbb{R}^n)} \rightarrow 0$ implies pointwise convergence almost everywhere on a subsequence, $v = u$ almost everywhere follows by uniqueness of limits. Passing to the limit $k \rightarrow \infty$ in $\|u_k\|_{L^\infty(\mathbb{R}^n)} \leq \|u_k\|_{W^{n,1}(\mathbb{R}^n)}$ proves the claim.

7.5. A variant of the Poincaré inequality

Let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded of class C^1 . Let μ be the Lebesgue measure on Ω . Let $1 \leq p < \infty$ and $\alpha > 0$. Towards a contradiction, we assume that there exists a sequence $(u_k)_{k \in \mathbb{N}}$ in $W^{1,p}(\Omega)$ such that for every $k \in \mathbb{N}$

$$\mu(\{x \in \Omega : u_k(x) = 0\}) \geq \alpha, \quad \|u_k\|_{L^p(\Omega)} > k \|\nabla u_k\|_{L^p(\Omega)}. \quad (5)$$

Without loss of generality, we may assume $\|u_k\|_{L^p(\Omega)} = 1$ for every $k \in \mathbb{N}$. Otherwise we replace u_k with $\|u_k\|_{L^p(\Omega)}^{-1} u_k$ which preserves both inequalities (5). As a consequence, $\|u_k\|_{W^{1,p}(\Omega)} < 1 + \frac{1}{k}$ for any $k \in \mathbb{N}$ which shows that $(u_k)_{k \in \mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$.

The Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact for any $1 \leq p < \infty$. In the case $1 \leq p < n$, compactness holds because $p < p^*$. In the case $n \leq p < \infty$ compactness holds because $W^{1,p}(\Omega) \hookrightarrow W^{1,n}(\Omega) \hookrightarrow L^p(\Omega)$, where the second embedding is compact (Korollar 8.5.1) and the first embedding continuous by Hölder's inequality.

Hence, there exists a subsequence $(u_k)_{k \in \Lambda \subset \mathbb{N}}$ and some $v \in L^p(\Omega)$ such that

$$\lim_{\Lambda \ni k \rightarrow \infty} \|u_k - v\|_{L^p(\Omega)} = 0.$$

Moreover, convergence in L^p implies that there exists a subsequence $(u_k)_{k \in \Lambda' \subset \Lambda}$ such that $u_k(x) \rightarrow v(x)$ converges pointwise for almost every $x \in \Omega$ as $\Lambda' \ni k \rightarrow \infty$. Since $\|\nabla u_k\|_{L^p(\Omega)} \rightarrow 0$ as $k \rightarrow \infty$ by (5) and since the space $W^{1,p}(\Omega)$ is complete, we have $v \in W^{1,p}(\Omega)$ satisfying $\nabla v = 0$ which according to problem 7.2 implies that v has a constant representative. (Here it is crucial that Ω is connected.) If we prove

$$\mu(\{x \in \Omega : v(x) = 0\}) \geq \alpha > 0,$$

then $v \equiv 0$ would follow which would contradict $\forall k \in \mathbb{N} : \|u_k\|_{L^p(\Omega)} = 1$. Let

$$A_m := \bigcup_{k \in \Lambda', k \geq m} \{x \in \Omega : u_k(x) = 0\},$$

$$A := \bigcap_{m=1}^{\infty} A_m.$$

Then, $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ and since $\mu(A_1) \leq \mu(\Omega) < \infty$ and $\mu(A_m) \geq \alpha$ we have

$$\mu(A) = \lim_{m \rightarrow \infty} \mu(A_m) \geq \alpha.$$

Since we have pointwise convergence $u_k(x) \rightarrow v(x)$ as $\Lambda' \ni k \rightarrow \infty$ for almost every $x \in A$ and since by construction, $u_k(x) = 0$ for infinitely many $k \in \Lambda'$ and every $x \in A$, we conclude $v(x) = 0$ for almost every $x \in A$. Therefore,

$$\mu(\{x \in \Omega : v(x) = 0\}) \geq \mu(A) \geq \alpha.$$

7.6. Explosion of the Poincaré constant

For $k \in \mathbb{N}$ let $\Omega_k = Q_+ \cup A_k \cup Q_-$ and $u: \Omega_k \rightarrow \mathbb{R}$ be given by

$$\begin{aligned} Q_+ &=]1, 3[\times]-1, 1[, \\ A_k &= [-1, 1] \times]-\frac{1}{k}, \frac{1}{k}[, \\ Q_- &=]-3, -1[\times]-1, 1[, \end{aligned} \quad u(x_1, x_2) = \begin{cases} 1, & \text{if } (x_1, x_2) \in Q_+, \\ x_1, & \text{if } (x_1, x_2) \in A_k, \\ -1, & \text{if } (x_1, x_2) \in Q_-. \end{cases}$$

Since u is Lipschitz continuous, $u \in W^{1,\infty}(\Omega)$ and because Ω is bounded $u \in W^{1,p}(\Omega)$ for any $1 \leq p < \infty$. Moreover, $u_{\Omega_k} = \int_{\Omega_k} u \, dx = 0$ and

$$\int_{\Omega_k} |u - u_{\Omega_k}|^p \, dx = \int_{\Omega_k} |u|^p \, dx \geq 8, \quad \int_{\Omega_k} |\nabla u|^p \, dx = \int_{A_k} 1 \, dx = \frac{4}{k}.$$

Combining these two facts with the assumed Poincaré inequality, we have $8 \leq C(\Omega_k) \frac{4}{k}$. Therefore, $C(\Omega_k) \geq 2k \rightarrow \infty$ as $k \rightarrow \infty$.

