Part I. Multiple choice questions

8.1. Let $\Omega \subset \mathbb{R}^3$ be open and bounded of class C^1. Into which space does $H^1(\Omega)$ *not* embed continuously?

√ (a) $C^0(\overline{\Omega})$

(b) $L^4(\Omega)$

(c) $L^6(\Omega)$

(d) $W^{1,1}(\Omega)$

(e) None of the above.

Since $\Omega \subset \mathbb{R}^3$ is bounded of class C^1, Sobolev's embedding theorem (Satz 8.5.3.) states that $H^1(\Omega) = W^{1,2}(\Omega)$ embeds into $L^q(\Omega)$ for every $1 \leq q \leq 6$. Moreover, since Ω is bounded, $\|u\|_{L^1} \leq C\|u\|_{L^2}$ and $\|\nabla u\|_{L^1} \leq C\|\nabla u\|_{L^2}$ for every $u \in H^1(\Omega)$. Hence $H^1(\Omega) \hookrightarrow W^{1,1}(\Omega)$. However, unlike in $C^0(\overline{\Omega})$, not every $u \in H^1(\Omega)$ is bounded.

8.2. Let $n \geq 2$ and $p = 2n$. Into which space does $W^{1,p}(\mathbb{R}^n)$ *not* embed continuously?

(a) $L^\infty(\mathbb{R}^n)$

(b) $\mathcal{L}^{p,p}(\mathbb{R}^n)$

√ (c) $L^n(\mathbb{R}^n)$

(d) $C^{0,\frac{1}{2}}(\mathbb{R}^n)$

(e) None of the above.

• According to Sobolev's embedding theorem (Satz 8.6.3), $W^{1,2n}(\mathbb{R}^n) \hookrightarrow C^{0,\frac{1}{2}}(\mathbb{R}^n)$. Functions $u \in C^{0,\frac{1}{2}}(\mathbb{R}^n)$ satisfy $\|u\|_{C^0(\mathbb{R}^n)} < \infty$, hence $u \in L^\infty(\mathbb{R}^n)$.

• According to the Poincaré-inequality (Satz 8.6.6), we have $W^{1,p}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{p,p}(\mathbb{R}^n)$.

• We claim that there exists $u \in W^{1,2n}(\mathbb{R}^n)$ such that $u \notin L^n(\mathbb{R}^n)$. Indeed, let $u \in C^\infty(\mathbb{R}^n)$ be given by $u(x) = |x|^{-\beta}$ whenever $|x| > 1$. Then, if $\frac{1}{2} < \beta < 1$, we have

$$\begin{aligned} 2\beta n > n &\Rightarrow \quad u \in L^{2n}(\mathbb{R}^n), \\ 2(\beta+1)n > n &\Rightarrow \nabla u \in L^{2n}(\mathbb{R}^n), \end{aligned} \qquad \beta n < n \Rightarrow u \notin L^n(\mathbb{R}^n).$$

8.3. Let $n \geq 2$ and $1 < p < n$. Let $\Omega \subset \mathbb{R}^n$ be bounded of class C^1. Let $u \in W^{1,p}(\Omega)$. Which statement is false?

(a) $u|_{\partial\Omega} \in L^p(\partial\Omega)$ is well-defined.

(b) The embedding $W^{1,p}(\Omega) \hookrightarrow L^{\frac{n}{n-p}}(\Omega)$ is compact.

√ (c) There exists $C < \infty$ independently of u such that $\|u\|_{W^{1,p}(\Omega)} \leq C\|u\|_{L^{\frac{np}{n-p}}(\Omega)}$.

(d) There exists more than one $v \in W^{1,p}(\mathbb{R}^n)$ with $v|_\Omega = u$.

(e) None of the above.

• According to the trace theorem (Satz 8.4.3), $u \mapsto u|_{\partial\Omega}$ defines a continuous map $W^{1,p}(\Omega) \to L^p(\partial\Omega)$ because $\partial\Omega$ is compact of class C^1.

• As $1 < p < n \Rightarrow \frac{n}{n-p} < \frac{np}{n-p} = p^*$, the embedding $W^{1,p}(\Omega) \hookrightarrow L^{\frac{n}{n-p}}(\Omega)$ is compact.

• The Sobolev inequality states $\|u\|_{L^{\frac{np}{n-p}}} \leq C\|u\|_{W^{1,p}(\Omega)}$. However, the reverse inequality fails to hold: Suppose $B_1 \subset \Omega$ and let $u_k(x) = (1 - |x|^k)_+$. Then, u is Lipschitz and hence in $W^{1,p}(\Omega)$. Moreover, $\|u_k\|_{L^{\frac{np}{n-p}}} \leq |B_1|^{\frac{n-p}{np}} \leq C$ and

$$\int_\Omega |\nabla u_k|^p \, dx \geq C' \int_0^1 \Big(kr^{k-1}\Big)^p r^{n-1} \, dr = C'k^p \int_0^1 r^{kp-p+n-1} \, dr = \frac{C'k^p}{kp-p+n-1}.$$

Since $p > 1$ we conclude $\|\nabla u_k\|_{L^p(\Omega)} \to \infty$ while $\|u_k\|_{L^{\frac{np}{n-2}}(\Omega)}$ stays bounded.

• The extension theorem (Satz 8.4.1) states that $u \in W^{1,p}(\Omega)$ can be extended to $Eu \in W^{1,p}(\mathbb{R}^n)$. Some perturbation of Eu outside Ω shows that the extension is not unique.

8.4. Let $n \geq 3$ and let $B_1 \subset \mathbb{R}^n$ be the unit ball. For which $q \geq 1$ is the following inequality true?

$$\exists C < \infty \quad \forall u \in C_c^\infty(B_1): \quad \int_{B_1} |u|^3 \, dx \leq C \Big(\int_{B_1} |\nabla u|^2 \, dx \Big) \Big(\int_{B_1} |u|^q \, dx \Big)^{\frac{2}{n}}$$

(a) any $q \geq 3$

(b) only for $q = 3$

(c) only for $q = \frac{n}{3}$

√ (d) only for $q = \frac{n}{2}$

(e) None of the above.

Suppose the inequality holds for some $q \in \mathbb{R}$. Replacing u by λu for $\lambda > 0$ we obtain

$$\lambda^3 \int_{B_1} |u|^3 \, dx \leq C \lambda^{2+q\frac{2}{n}} \Big(\int_{B_1} |\nabla u|^2 \, dx \Big) \Big(\int_{B_1} |u|^q \, dx \Big)^{\frac{2}{n}}.$$

If $q < \frac{n}{2}$, then $\lambda \to \infty$ leads to a contradiction. If $q > \frac{n}{2}$, then $\lambda \to 0$ leads to a contradiction. (The powers of λ must coincide left and right.) The inequality actually holds with $q = \frac{n}{2}$, since we can estimate

$$\begin{aligned} \int_{B_1} |u|^3 \, dx &= \int_{B_1} |u|^2 |u| \, dx \\ &\leq \Big(\int_{B_1} |u|^{\frac{2n}{n-2}} \, dx \Big)^{\frac{n-2}{n}} \Big(\int_{B_1} |u|^{\frac{n}{2}} \, dx \Big)^{\frac{2}{n}} && \text{(Hölder)} \\ &\leq C \Big(\int_{B_1} |\nabla u|^2 \, dx \Big) \Big(\int_{B_1} |u|^{\frac{n}{2}} \, dx \Big)^{\frac{2}{n}}. && \text{(Sobolev)} \end{aligned}$$

8.5. Let $n \in \mathbb{N}$ and let $B_R \subset \mathbb{R}^n$ be the ball of radius $R > 0$ around the origin. Let $1 \leq p < n$ and let $1 \leq q \leq \frac{np}{n-p}$. For which $\beta \in \mathbb{R}$ is the following statement true?

$$\exists C < \infty \quad \forall R > 0 \quad \forall u \in W_0^{1,p}(B_R): \qquad \|u\|_{L^q(B_R)} \leq C R^{\beta} \|\nabla u\|_{L^p(B_R)}$$

(a) $\beta = 0$

(b) $\beta = 1$

(c) $\beta = \frac{n-p}{q}$

√ (d) $\beta = \frac{n}{q} - \frac{n}{p} + 1$

(e) None of the above.

Given $u \in W_0^{1,p}(B_R)$ let $\tilde{u} \in W_0^{1,p}(B_1)$ be defined by $\tilde{u}(y) = u(Ry)$. Then

$$\begin{aligned} \int_{B_R} |u(x)|^q \, dx &= \int_{B_1} |u(Ry)|^q R^n \, dy = R^n \int_{B_1} |\tilde{u}(y)|^q \, dy, \\ \Rightarrow \quad \|u\|_{L^q(B_R)} &= R^{\frac{n}{q}} \|\tilde{u}\|_{L^q(B_1)}, \end{aligned}$$

$$\begin{aligned} \int_{B_R} |(\nabla u)(x)|^p \, dx &= \int_{B_1} |(\nabla u)(Ry)|^p R^n \, dy \\ &= \int_{B_1} \Big|\nabla(u(Ry)) \tfrac{1}{R}\Big|^p R^n \, dy = R^{n-p} \int_{B_1} |\nabla \tilde{u}|^p \, dy, \\ \Rightarrow \quad \|\nabla u\|_{L^p(B_R)} &= R^{\frac{n}{p}-1} \|\nabla \tilde{u}\|_{L^p(B_1)}. \end{aligned}$$

The function $\tilde{u} \in W_0^{1,p}(B_1)$ can be extended by zero to $\overline{u} \in W^{1,p}(\mathbb{R}^n)$. Hence,

$$\|\tilde{u}\|_{L^q(B_1)} \leq C_1 \|\tilde{u}\|_{L^{p^*}(B_1)} = C_1 \|\overline{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|\nabla \overline{u}\|_{L^p(\mathbb{R}^n)} = C \|\nabla \tilde{u}\|_{L^p(B_1)}$$

follows from Hölder's inequality and the Sobolev inequality. Consequently,

$$\|u\|_{L^q(B_R)} = R^{\frac{n}{q}} \|\tilde{u}\|_{L^q(B_1)} \leq C R^{\frac{n}{q}} \|\nabla \tilde{u}\|_{L^p(B_1)} = C R^{\frac{n}{q} - \frac{n}{p} + 1} \|\nabla u\|_{L^p(B_1)}.$$

Moreover, $\beta = \frac{n}{q} - \frac{n}{p} + 1$ is the only possible choice. For any other β either $R \to 0$ or $R \to \infty$ leads to a contradiction.

Part II. True or false?

8.6. Let $u \in W^{1,1}(\mathbb{R}^n)$. Let $\Omega \subset \mathbb{R}^n$ be any open domain. Then $u|_\Omega \in W^{1,1}(\Omega)$.

√ (a) True.

(b) False.

By monotonicity of the integral with respect to the domain, $\|u\|_{L^1(\Omega)} \leq \|u\|_{L^1(\mathbb{R}^n)} < \infty$ and $\|\nabla u\|_{L^1(\Omega)} \leq \|\nabla u\|_{L^1(\mathbb{R}^n)} < \infty$.

8.7. Let $n \in \mathbb{N}$ and $1 \leq p < \infty$. The spaces $W^{1,p}(\mathbb{R}^n)$ and $W_0^{1,p}(\mathbb{R}^n)$ are the same.

√ (a) True.

(b) False.

For every $u \in W^{1,p}(\mathbb{R}^n)$ there exists a sequence of functions $u_k \in C_c^\infty(\mathbb{R}^n)$ such that $\|u_k - u\|_{W^{1,p}(\mathbb{R}^n)} \to 0$ as $k \to \infty$ (Korollar 8.4.2 with $\Omega = \mathbb{R}^n$). By definition, $W_0^{1,p}(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ in the $W^{1,p}$-norm. Therefore, $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$.

8.8. Let $n \geq 2$ and $1 \leq p < n$. The spaces $W^{1,p}(\mathbb{R}^n \setminus \overline{B_1(0)})$ and $W_0^{1,p}(\mathbb{R}^n \setminus \overline{B_1(0)})$ are the same.

(a) True.

√ (b) False.

Since the domain $\Omega := \mathbb{R}^n \setminus \overline{B_1(0)}$ is open of class C^1 with compact boundary, the trace operator $T\colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is well-defined and continuous. Let $u \in W_0^{1,p}(\Omega)$. By definition of $W_0^{1,p}(\Omega)$, there exists a sequence $(u_k)_{k\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ such that $\|u_k - u\|_{W^{1,p}(\mathbb{R}^n)} \to 0$ as $k \to \infty$. By linearity and continuity of the trace operator, $\|Tu_k - Tu\|_{L^p(\partial\Omega)} \to 0$ as $k \to \infty$. Since $Tu_k \equiv 0$ by definition of $C_c^\infty(\Omega)$, we have $u|_{\partial\Omega} = Tu = 0$ in $L^p(\partial\Omega)$. However, not every function $v \in W^{1,p}(\Omega)$ satisfies $Tv = 0$. Therefore $W_0^{1,p}(\Omega) \neq W^{1,p}(\Omega)$.

8.9. Let $n \geq 2$ and $1 \leq p < n$. The spaces $W^{1,p}(\mathbb{R}^n \setminus \{0\})$ and $W_0^{1,p}(\mathbb{R}^n \setminus \{0\})$ are the same.

√ (a) True.

(b) False.

Since $\{0\} \subset \mathbb{R}^n$ has vanishing $W^{1,n}$-capacity (Beispiel 8.1.1), there exists a sequence $(\psi_k)_{k\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^n)$ satisfying $0 \leq \psi_k \leq 1$ and $\psi_k = 1$ in a neighbourhood of $\{0\}$ such that $\psi_k \to 0$ almost everywhere and $\|\nabla\psi_k\|_{L^n} \to 0$ as $k \to \infty$.

Given $u \in W^{1,p}(\mathbb{R}^n \setminus \{0\})$ let $u_k \in C_c^\infty(\mathbb{R}^n)$ such that $\|u - u_k\|_{W^{1,p}(\mathbb{R}^n\setminus\{0\})} \to 0$ (Korollar 8.4.2). Let $v_k = (1 - \psi_k)u_k$. Then, $v_k \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$. The triangle inequality, Hölder's estimate and the Sobolev inequality for $p < n$ yield

$$\|u - v_k\|_{W^{1,p}} \leq \|u - u_k\|_{W^{1,p}} + \|\psi_k u_k\|_{W^{1,p}}, \tag{I$_k$}$$

$$\|\psi_k u_k\|_{W^{1,p}} \leq \|\psi_k u_k\|_{L^p} + \|\psi_k \nabla u_k\|_{L^p} + \|u_k \nabla\psi_k\|_{L^p}, \tag{II$_k$}$$

$$\|u_k \nabla\psi_k\|_{L^p} \leq \|u_k\|_{L^{p^*}} \|\nabla\psi_k\|_{L^n} \leq \|u_k\|_{W^{1,p}} \|\nabla\psi_k\|_{L^n}. \tag{III$_k$}$$

Since $(u_k)_{k\in\mathbb{N}}$ is convergent in $W^{1,p}$, we have $\|u_k\|_{W^{1,p}} \leq C$ for some constant $C < \infty$. Hence, $(\mathrm{III}_k) \to 0$ as $k \to \infty$. Moreover, since $\|\psi_k u_k\|_{L^p} + \|\psi_k \nabla u_k\|_{L^p} \leq \|u_k\|_{W^{1,p}} \leq C$ and since $\psi_k u_k \to 0$ and $\psi_k \nabla u_k \to 0$ pointwise almost everywhere, the dominated convergence theorem applies and we obtain first $(\mathrm{II}_k) \to 0$ and then $(\mathrm{I}_k) \to 0$ as $k \to \infty$. Therefore, $u \in W_0^{1,p}(\mathbb{R}^n \setminus \{0\})$ and we conclude $W^{1,p}(\mathbb{R}^n \setminus \{0\}) = W_0^{1,p}(\mathbb{R}^n \setminus \{0\})$.

8.10. Let $n = 1$ and $p = 1$. The spaces $W^{1,1}(\mathbb{R} \setminus \{0\})$ and $W_0^{1,1}(\mathbb{R} \setminus \{0\})$ are the same.

(a) True.

√ (b) False.

Let $v \in W^{1,1}(\mathbb{R} \setminus \{0\})$ be arbitrary. Then, v is continuous and for any $x \in]0,1[$

$$\begin{aligned} v(0) &= v(x) - \int_0^x v' \, ds, \\ \Rightarrow \ |v(0)| &= \int_0^1 |v(0)| \, dx \leq \int_0^1 |v(x)| \, dx + \int_0^1 |v'| \, dx = \|v\|_{W^{1,1}(]0,1[)}. \end{aligned}$$

Let $u \in W_0^{1,1}(\mathbb{R} \setminus \{0\})$ and $u_k \in C_c^\infty(\mathbb{R} \setminus \{0\})$ such that $\|u - u_k\|_{W^{1,1}} \to 0$ as $k \to \infty$. Then, choosing $v = u - u_k$ in the estimate above, we obtain

$$|u(0)| = |u(0) - u_k(0)| \leq \|u - u_k\|_{W^{1,1}(]0,1[)} \xrightarrow{k \to \infty} 0.$$

Hence $u(0) = 0$. However, not every function $w \in W^{1,1}(\mathbb{R} \setminus \{0\})$ has this property. Therefore, $W_0^{1,1}(\mathbb{R}^n \setminus \{0\}) \neq W^{1,1}(\mathbb{R}^n \setminus \{0\})$.

8.11. Let $\Omega \subset \mathbb{R}^n$ be bounded of class C^1. Then, $C^\infty(\overline{\Omega})$ is dense in $C^{0,\frac{1}{2}}(\overline{\Omega})$.

(a) True.

√ (b) False.

See Bemerkung 8.6.2 in the notes.

8.12. Let $B_1 \subset \mathbb{R}^2$ be the unit ball and let $(u_k)_{k \in \mathbb{N}}$ be a bounded sequence in $W^{1,4}(B_1)$. Then, there exists $v \in C^{0,\frac{1}{4}}(\overline{B_1})$ and $\Lambda \subset \mathbb{N}$ such that $\|v - u_k\|_{C^{0,\frac{1}{4}}(\overline{B_1})} \to 0$ as $\Lambda \ni k \to \infty$.

√ (a) True.

(b) False.

Since $B_1 \subset \mathbb{R}^2$ is bounded of class C^1, the embedding $W^{1,4}(B_1) \hookrightarrow C^{0,\alpha}(\overline{B_1})$ is compact for every $0 \leq \alpha < 1 - \frac{2}{4} = \frac{1}{2}$, in particular for $\alpha = \frac{1}{4}$.

8.13. There exists functions in $W^{1,3}(\mathbb{R}^2)$ such that all their representatives are nowhere differentiable.

(a) True.

√ (b) False.

According to Satz 8.6.10, the continuous representative of $u \in W^{1,3}(\mathbb{R}^2)$ is differentiable classically almost everywhere.

8.14. Every compactly supported function in $W^{1,4}(\mathbb{R}^3)$ is in $L^\infty(\mathbb{R}^3)$.

√ (a) True.

(b) False.

According to Sobolev's embedding $W^{1,4}(\mathbb{R}^3) \hookrightarrow C^{0,\alpha}(\mathbb{R}^3)$ with $\alpha = 1 - \frac{3}{4} = \frac{1}{4}$, every $u \in W^{1,4}(\mathbb{R}^3)$ is Hölder continuous and $\|u\|_{C^{0,\frac{1}{4}}(\mathbb{R}^3)} \leq C\|u\|_{W^{1,4}(\mathbb{R}^3)}$. The claim follows from $\|u\|_{L^\infty} \leq \|u\|_{C^0} \leq \|u\|_{C^{0,\frac{1}{4}}}$. (Compact support is not needed here.)

8.15. Let $B_1 \subset \mathbb{R}^n$ be the unit ball. There exists a sequence $(u_k)_{k\in\mathbb{N}}$ in $C_c^\infty(B_1)$ satisfying $u_k(x) = 1$ for every $x \in B_1$ with $|x| = \frac{1}{2}$ and $\|u_k\|_{W^{1,1}(B_1)} \to 0$ as $k \to \infty$.

(a) True.

√ (b) False.

Since $v_k := u_k|_{B_{\frac{1}{2}}} \in W^{1,1}(B_{\frac{1}{2}})$ the trace theorem (Satz 8.4.3) implies that

$$\|u_k\|_{L^1(\partial B_{\frac{1}{2}})} = \|v_k\|_{L^1(\partial B_{\frac{1}{2}})} \leq C\|v_k\|_{W^{1,1}(B_{\frac{1}{2}})} \leq C\|u_k\|_{W^{1,1}(B_1)}.$$

Hence, if $u_k = 1$ on $\partial B_{\frac{1}{2}}$, then $\|u_k\|_{W^{1,1}(B_1)}$ cannot converge to zero.