9.1. Elliptic equations in non-divergence form

(a) Given $u, \varphi \in H_0^1(\Omega)$ let

$$B(u,\varphi) = \sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j}\varphi \, dx + \int_\Omega c u \varphi \, dx.$$

Then, with some unspecified constant $C < \infty$ depending only on a_{ij}, c and Ω,

$$\begin{aligned}
&|B(u,\varphi)| \\
&\leq \sum_{i,j=1}^n \int_\Omega |a_{ij}||\nabla u||\nabla\varphi| + |\nabla a_{ij}||\nabla u||\varphi| + |c||u||\varphi| \, dx \\
&\leq \sum_{i,j=1}^n \Big(\|a_{ij}\|_{C^0}\|\nabla u\|_{L^2}\|\nabla\varphi\|_{L^2} + \|\nabla a_{ij}\|_{C^0}\|\nabla u\|_{L^2}\|\varphi\|_{L^2} + \|c\|_{C^0}\|u\|_{L^2}\|\varphi\|_{L^2}\Big) \\
&\leq C\|\nabla u\|_{L^2}\|\nabla\varphi\|_{L^2},
\end{aligned}$$

where we used that $\|a_{ij}\|_{C^1(\Omega)}$ and $\|c\|_{C^0}$ are finite for every $i,j \in \{1,\ldots,n\}$ and applied the Poincaré inequality: $\|\varphi\|_{L^2(\Omega)} \leq \tilde{C}\|\nabla\varphi\|_{L^2(\Omega)}$ and $\|u\|_{L^2(\Omega)} \leq \tilde{C}\|\nabla u\|_{L^2(\Omega)}$.
Moreover, ellipticity implies the following lower bound for the first term of $B(u,u)$.

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\, dx \geq \int_\Omega \lambda|\nabla u|^2 \, dx = \lambda\|\nabla u\|^2_{L^2(\Omega)}.$$

Since $u \in H_0^1(\Omega)$ we may integrate the second term of $B(u,u)$ by parts with vanishing boundary terms to obtain

$$\sum_{i,j=1}^n \int_\Omega \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} u \, dx = \frac{1}{2}\sum_{i,j=1}^n \int_\Omega \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u^2}{\partial x_j}\, dx = -\frac{1}{2}\sum_{i,j=1}^n \int_\Omega \frac{\partial^2 a_{ij}}{\partial x_j \partial x_i} u^2 \, dx.$$

Under the given assumptions on a_{ij} and c, we conclude

$$B(u,u) \geq \lambda\|\nabla u\|^2_{L^2(\Omega)}.$$

Therefore, the Lax-Milgram Lemma (Satz 4.3.3) applies to $B\colon H_0^1(\Omega)\times H_0^1(\Omega) \to \mathbb{R}$ and (by Korollar 4.3.1) we obtain a unique $u \in H_0^1(\Omega)$ such that

$$\forall \varphi \in H_0^1(\Omega): \quad B(u,\varphi) = \int f\varphi \, dx.$$

(b) If $u \in C^2(\Omega)\cap H_0^1(\Omega)$ is a classical solution of $-Lu + cu = f$ and $\varphi \in H_0^1(\Omega)$,

$$\begin{aligned}
\int_\Omega f\varphi \, dx &= -\sum_{i,j=1}^n \int_\Omega a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\varphi \, dx + \int_\Omega cu\varphi \, dx \\
&= \sum_{i,j=1}^n \int_\Omega a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j}\varphi \, dx + \int_\Omega cu\varphi \, dx = B(u,\varphi).
\end{aligned}$$

9.2. The reflection Lemma towards boundary regularity

Given $f \in L^2(\mathbb{R}^n_+)$ let $u \in H^1_0(\mathbb{R}^n_+)$ with $\operatorname{supp}(u) \subset\subset \mathbb{R}^n$ be a weak solution to

$$-\Delta u = f \quad \text{in } \mathbb{R}^n_+.$$

We introduce the notation

$$x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}, \qquad \nabla' u = \Big(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_{n-1}}, 0\Big)$$

and define

$$\overline{u}(x) = \begin{cases} u(x', x_n) & \text{if } x_n > 0, \\ -u(x', -x_n) & \text{if } x_n < 0, \end{cases} \qquad \overline{f}(x) = \begin{cases} f(x', x_n) & \text{if } x_n > 0, \\ -f(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

Since $u \in H^1_0(\mathbb{R}^n_+)$ has compact support in $\mathbb{R}^n$, the extension $\overline{u}$ via odd reflection is in $H^1(\mathbb{R}^n)$ (compare Problem 3.5). Let $\varphi \in C^\infty_c(\mathbb{R}^n)$ be arbitrary. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \nabla\overline{u} \cdot \nabla\varphi \, dx \\
&= \int_{\mathbb{R}^n_+} \nabla' u(x', x_n) \cdot \nabla'\varphi(x', x_n) - \nabla' u(x', x_n) \cdot \nabla'\varphi(x', -x_n) \, dx \\
&\quad + \int_{\mathbb{R}^n_+} \frac{\partial u}{\partial x_n}(x', x_n) \frac{\partial \varphi}{\partial x_n}(x', x_n) + \frac{\partial u}{\partial x_n}(x', x_n) \frac{\partial \varphi}{\partial x_n}(x', -x_n) \, dx \\
&= \int_{\mathbb{R}^n_+} \nabla u(x) \cdot \nabla\Big(\varphi(x', x_n) - \varphi(x', -x_n)\Big) \, dx \\
&= \int_{\mathbb{R}^n_+} \nabla u \cdot \nabla\psi \, dx,
\end{aligned}$$

where $\psi(x', x_n) = \varphi(x', x_n) - \varphi(x', -x_n)$ satisfies $\psi(x', 0) = 0$, i. e. $\psi \in C^\infty \cap H^1_0(\mathbb{R}^n_+)$. Because u is a weak solution to $-\Delta u = f$ in $\mathbb{R}^n_+$ by assumption, we have

$$\begin{aligned}
\int_{\mathbb{R}^n_+} \nabla u \cdot \nabla\psi \, dx &= \int_{\mathbb{R}^n_+} f\psi \, dx \\
&= \int_{\mathbb{R}^n_+} f\varphi \, dx - \int_{\mathbb{R}^n_+} f(x', x_n)\varphi(x', -x_n) \, dx \\
&= \int_{\mathbb{R}^n} \overline{f}\varphi \, dx.
\end{aligned}$$

Since φ is arbitrary, if follows that $\overline{u} \in H^1(\mathbb{R}^n)$ is a weak solution of $-\Delta\overline{u} = \overline{f}$ in $\mathbb{R}^n$.

9.3. Horizontal derivatives

Given $u \in H^2(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+)$ and $h \in \mathbb{R} \setminus \{0\}$, let $D_{h,i}u \colon \mathbb{R}^n_+ \to \mathbb{R}$ be given by

$$D_{h,i}u(x) = \frac{u(x+he_i) - u(x)}{h},$$

where $e_i = (0, \dots, 0, 1, 0 \dots, 0, 0) \in \mathbb{R}^n$ has the entry 1 at position $i \in \{1, \dots, n-1\}$.

The translation by he_i is an isometry of $H^1(\mathbb{R}^n_+)$ and carries $C^\infty_c(\mathbb{R}^n_+)$ into itself, so it carries its closure $H^1_0(\mathbb{R}^n_+)$into itself. Therefore, $u \in H^1_0(\Omega)$ implies $D_{h,i}u \in H^1_0(\mathbb{R}^n_+)$.

According to Satz 8.3.1.iii) the assumption $u \in H^2(\mathbb{R}^n_+)$ implies

$$\exists C < \infty \quad \forall h \in \mathbb{R}^n \setminus \{0\} : \quad \|D_{h,i}u\|_{H^1} \leq C.$$

Hence, there exists a sequence $h_k \xrightarrow{k\to\infty} 0$ such that $D_{h_k,i}u$ converges weakly in $H^1(\mathbb{R}^n_+)$ to some $v \in H^1(\mathbb{R}^n_+)$ as $k \to \infty$. Since $H^1_0(\mathbb{R}^n_+)$ is a closed subspace of $H^1(\mathbb{R}^n_+)$, it is weakly closed. Therefore, $v \in H^1_0(\mathbb{R}^n_+)$. Moreover, for any $\varphi \in C^\infty_c(\mathbb{R}^n_+)$ there holds

$$\begin{aligned}
\int_{\mathbb{R}^n_+} v\varphi \, dx &= \lim_{k\to\infty} \int_{\mathbb{R}^n_+} \frac{u(x+h_ke_i) - u(x)}{h_k} \varphi(x) \, dx \\
&= \lim_{k\to\infty} \frac{1}{h_k} \Big(\int_{\mathbb{R}^n_+} u(x+h_ke_i)\varphi(x) \, dx - \int_{\mathbb{R}^n_+} u(x)\varphi(x) \, dx \Big) \\
&= \lim_{k\to\infty} \frac{1}{h_k} \Big(\int_{\mathbb{R}^n_+} u(y)\varphi(y-h_ke_i) \, dy - \int_{\mathbb{R}^n_+} u(x)\varphi(x) \, dx \Big) \\
&= -\lim_{k\to\infty} \int_{\mathbb{R}^n_+} u(x) \frac{\varphi(x) - \varphi(x-h_ke_i)}{h_k} \, dx \\
&= -\int_{\mathbb{R}^n_+} u \frac{\partial \varphi}{\partial x_i} \, dx.
\end{aligned}$$

By definition of weak derivative,

$$\frac{\partial u}{\partial x_i} = v \in H^1_0(\mathbb{R}^n_+)$$

and the claim follows.

9.4. Properties of the bilaplacian

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary and

$$\Xi := \{u \in H^4(\Omega) \cap H_0^1(\Omega) : \Delta u \in H_0^1(\Omega)\}.$$

(a) Since the bilaplacian $\Delta^2 \colon \Xi \to L^2(\Omega)$ is linear, it suffices to prove $\ker(\Delta^2) = \{0\}$ to conclude injectivity. Let $u \in \Xi$ with $\Delta^2 u = 0$. By definition of Ξ, we have

$$v := \Delta u \in H^2(\Omega) \cap H_0^1(\Omega).$$

Moreover, $\Delta v = 0$ combined with the elliptic regularity estimate (Satz 9.1.2) implies $v = 0$. Repeating the same argument for $\Delta u = 0$ yields $u = 0$ and proves $\ker(\Delta^2) = 0$.

To prove surjectivity, let $f \in L^2(\Omega)$ be given arbitrarily. Let $v \in H_0^1(\Omega)$ be the weak solution to $\Delta v = f$ in Ω. By elliptic regularity, $v \in H^2(\Omega)$. Let $u \in H_0^1(\Omega)$ be the weak solution to $\Delta u = v$. Then, by elliptic regularity, $u \in H^4(\Omega)$. Consequently, $u \in \Xi$. Since $\Delta^2 u = f$ by construction, surjectivity of $\Delta^2 \colon \Xi \to L^2(\Omega)$ follows.

(b) Given $f \in L^2(\Omega)$, let $u \in \Xi$ satisfy $\Delta^2 u = f$. Let $\varphi \in \Xi$ be arbitrary. Then, $\nabla\Delta\varphi \in L^2(\Omega)$. Since $u \in H_0^1(\Omega)$, the trace theroem (Satz 8.4.3) implies that $u|_{\partial\Omega} \in L^2(\partial\Omega)$ is well-defined and vanishes according to Korollar 8.4.3. Analogously, since $\Delta\varphi \in H_0^1(\Omega)$ by assumption, $(\Delta v)|_{\partial\Omega} = 0$. Hence, we may integrate by parts twice with vanishing boundary terms to obtain

$$\int_\Omega u\Delta^2\varphi\,dx = -\int_\Omega \nabla u \cdot \nabla\Delta\varphi\,dx = \int_\Omega \Delta u \Delta\varphi\,dx. \tag{$*$}$$

Since the right hand side of $(*)$ is symmetric in u and φ we may switch the roles of $u, \varphi \in \Xi$ to also obtain

$$\int_\Omega \varphi\Delta^2 u\,dx = \int_\Omega \Delta u\Delta\varphi\,dx = \int_\Omega u\Delta^2\varphi\,dx.$$

Since $\varphi \in \Xi$ is arbitrary, the claim follows by substituting $\Delta^2 u = f$.

(c) Given $f \in L^2(\Omega)$, let $u \in L^2(\Omega)$ satisfy

$$\forall\varphi \in \Xi : \quad \int_\Omega u\Delta^2\varphi\,dx = \int_\Omega f\varphi\,dx. \tag{$\dagger$}$$

According to part (a), there exists $v \in \Xi$ such that $\Delta^2 v = f$. Moreover, by part (b)

$$\forall\varphi \in \Xi : \quad \int_\Omega v\Delta^2\varphi\,dx = \int_\Omega f\varphi\,dx.$$

Therefore, using again bijectivity of $\Delta^2 \colon \Xi \to L^2(\Omega)$ as shown in (a), we have

$$\forall\varphi \in \Xi : \quad \int_\Omega (u - v)\Delta^2\varphi\,dx = 0 \quad \overset{(a)}{\iff} \quad \forall\psi \in L^2(\Omega) : \quad \int_\Omega (u - v)\psi\,dx = 0.$$

Hence $u - v = 0$ in $L^2(\Omega)$. Therefore, $u = v \in \Xi$ as claimed.

9.5. Weak solutions to the bilaplace equation

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary.

(a) The map $\langle\cdot,\cdot\rangle \colon \big(H^2(\Omega)\cap H^1_0(\Omega)\big)\times\big(H^2(\Omega)\cap H^1_0(\Omega)\big)\to\mathbb{R}$ given by

$$\langle u,v\rangle := \int_\Omega \Delta u \Delta v\,dx$$

is symmetric and bilinear by definition. Moreover, by the elliptic regularity estimate (Satz 9.1.2), there exists a constant $C<\infty$ such that for every $u\in H^2(\Omega)\cap H^1_0(\Omega)$

$$\langle u,u\rangle \le (u,u)_{H^2(\Omega)} = \|u\|^2_{H^2(\Omega)} \le C\|\Delta u\|^2_{L^2(\Omega)} = C\langle u,u\rangle.$$

In particular, $\langle u,u\rangle\ge 0$ and $\langle u,u\rangle = 0 \Leftrightarrow u=0$; hence $\langle\cdot,\cdot\rangle$ defines a scalar product and $\langle\cdot,\cdot\rangle$ is equivalent to $(\cdot,\cdot)_{H^2(\Omega)}$.

(b) Since Ω is bounded, convergence in $H^2(\Omega)$ implies convergence in $H^1(\Omega)$. Since $H^1_0(\Omega)$ is closed in $H^1(\Omega)$, we obtain that $H^2(\Omega)\cap H^1_0(\Omega)$ is closed in $H^2(\Omega)$. Hence, $(H^2(\Omega)\cap H^1_0(\Omega),\langle\cdot,\cdot\rangle)$ is a Hilbert space.

(c) Let $f\in L^2(\Omega)$. Then the map $H^2(\Omega)\cap H^1_0(\Omega)\to\mathbb{R}$ given by $v\mapsto\int_\Omega fv\,dx$ is a continuous linear functional. By part (b) we may apply the Riesz representation theorem to conclude that there exists a unique $u\in H^2(\Omega)\cap H^1_0(\Omega)$ satisfying

$$\forall v\in H^2(\Omega)\cap H^1_0(\Omega):\quad \langle u,v\rangle = \int_\Omega fv\,dx.$$

In particular, for any $v\in\Xi := \{u\in H^4(\Omega)\cap H^1_0(\Omega) : \Delta u\in H^1_0(\Omega)\}$,

$$\int_\Omega u\Delta^2 v\,dx = \int_\Omega \Delta u\Delta v = \int fv\,dx.$$

Hence, $u\in\Xi$ according to problem 9.4 (c) and

$$\int_\Omega (\Delta^2 u)v\,dx = \int_\Omega u\Delta^2 v\,dx = \int fv\,dx$$

for any $v\in C^\infty_c(\Omega)$ which implies $\Delta^2 u = f$.