Part I. Multiple choice questions

10.1. Let $B_1 \subset \mathbb{R}^4$ be the unit ball and let $f \in H^k(B_1)$. Let $u \in H_0^1(B_1)$ be a weak solution of $-\Delta u = f$ in B_1. What is the minimal value of $k \in \mathbb{N}$ that guarantees u to be a classical solution in $C^2(\overline{B_1})$?

(a) $k = 1$

(b) $k = 2$

√ (c) $k = 3$

(d) $k = 4$

(e) None of the above.

By elliptic regularity, $u \in H^{k+2}(B_1)$. With $p = 2$ and $n = 4$ we have $k+2-\frac{n}{p} = k \in \mathbb{N}$. Hence, $u \in C^2(\overline{B_1})$ if $k = 2 + 1 = 3$.

10.2. Let $B_1 \subset \mathbb{R}^n$ be the unit ball and let $f \in H^k(B_1)$ for some $k \in \mathbb{N}$. Let $u \in H_0^1(B_1)$ be a weak solution of $-\Delta u = f$ in B_1. Then u is always bounded in B_1 provided

(a) $n > \frac{k}{2} + 1$

(b) $n > 2k$

(c) $n < 4k + 2$

√ (d) $n < 2k + 4$

(e) None of the above.

By elliptic regularity, $u \in H^{k+2}(B_1)$. This space embeds into $C^0(\overline{\Omega})$ if $2(k+2) > n$.

10.3. Given $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a weak solution of $-\Delta u = f$ in $\mathbb{R}^n$. For what $\alpha, \beta \in \mathbb{R}$ is the following statement true? (with C independent of u)

$$\exists C < \infty \quad \forall R > 0: \quad \int_{B_R} |\nabla u|^2 \, dx \leq C\Big(R^\alpha \int_{B_{2R}} |u|^2 \, dx + R^\beta \int_{B_{2R}} |f|^2 \, dx\Big).$$

(a) $\alpha = 2$ and $\beta = 2$

√ (b) $\alpha = -2$ and $\beta = 2$

(c) $\alpha = 2$ and $\beta = -2$

(d) $\alpha = -2$ and $\beta = -2$

(e) None of the above.

By elliptic regularity, we have $u \in H^2(B_{2R})$. Let $\phi \in C^\infty_c(B_{2R})$ satisfy $0 \leq \phi \leq 1$ and $\phi|_{B_R} \equiv 1$ as well as $|\nabla \phi| < \frac{2}{R}$. Then, $\varphi = \phi^2$ satisfies $|\nabla\varphi|^2 = 4\phi^2 |\nabla\phi|^2 \leq \frac{16}{R^2}\varphi$ and

$$\begin{aligned}
&\int_{B_{2R}} |\nabla u|^2 \varphi \, dx \\
&= -\int_{B_{2R}} u \nabla u \cdot \nabla\varphi \, dx - \int_{B_{2R}} u \varphi \Delta u \, dx \\
&\leq \Big(\int_{B_{2R}} |u|^2 \frac{|\nabla\varphi|^2}{\varphi} \, dx\Big)^{\frac{1}{2}} \Big(\int_{B_{2R}} |\nabla u|^2 \varphi \, dx\Big)^{\frac{1}{2}} + \Big(\int_{B_{2R}} |u|^2 \, dx\Big)^{\frac{1}{2}} \Big(\int_{B_{2R}} \varphi^2 |f|^2 \, dx\Big)^{\frac{1}{2}} \\
&\leq \frac{16}{2R^2} \int_{B_{2R}} |u|^2 \, dx + \frac{1}{2} \int_{B_{2R}} |\nabla u|^2 \varphi \, dx + \frac{1}{2R^2} \int_{B_{2R}} |u|^2 \, dx + \frac{R^2}{2} \int_{B_{2R}} |f|^2 \, dx.
\end{aligned}$$

Hence, by absorbing the gradient term,

$$\int_{B_R} |\nabla u|^2 \, dx \leq \int_{B_{2R}} |\nabla u|^2 \varphi \, dx \leq 17 R^{-2} \int_{B_{2R}} |u|^2 \, dx + R^2 \int_{B_{2R}} |f|^2 \, dx.$$

For any $\alpha \neq -2$ we can construct a counterexample. Let $u(x_1, x_2) = x_1$. Then $\Delta u = 0 =: f$ and $|\nabla u| = 1$. Moreover, for any $R > 0$

$$R^\alpha \int_{B_{2R}} |u|^2 \, dx \leq R^\alpha (2R)^{n-1} \int_{-2R}^{2R} x_1^2 \, dx_1 = \frac{2}{3} R^\alpha (2R)^{n+2} \leq c_n |B_R| R^{\alpha+2}.$$

Therefore, the statement requires $C \geq \frac{1}{c_n} R^{-2-\alpha}$ which blows up as $R \to 0$ if $\alpha > -2$ or as $R \to \infty$ if $\alpha < -2$. Hence, $\alpha = -2$ is the only possible choice.

Suppose the statement also holds with $\alpha = -2$ and some $\beta \neq 2$. Given $u \in C^2(\mathbb{R}^n)$, let $\tilde{u}(y) = u(\frac{y}{R})$. Then,

$$\begin{aligned}
\int_{B_2} |u(x)|^2 \, dx &= \int_{B_{2R}} |u(\tfrac{y}{R})|^2 R^{-n} \, dy = R^{-n} \int_{B_{2R}} |\tilde{u}(y)|^2 \, dy, \\
\int_{B_1} |(\nabla u)(x)|^2 \, dx &= \int_{B_R} |(\nabla u)(\tfrac{y}{R})|^2 R^{-n} \, dy = R^{2-n} \int_{B_R} |\nabla \tilde{u}(y)|^2 \, dy, \\
\int_{B_2} |(\Delta u)(x)|^2 \, dx &= \int_{B_{2R}} |(\Delta u)(\tfrac{y}{R})|^2 R^{-n} \, dy = R^{4-n} \int_{B_{2R}} |\Delta \tilde{u}(y)|^2 \, dy.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\int_{B_1} |\nabla u|^2 \, dx = R^{2-n} \int_{B_R} |\nabla \tilde{u}|^2 \, dy &\leq C R^{-n} \int_{B_{2R}} |\tilde{u}|^2 \, dy + C R^{\beta+2-n} \int_{B_{2R}} |\Delta \tilde{u}|^2 \, dx \\
&= C \int_{B_2} |u|^2 \, dy + C R^{\beta-2} \int_{B_2} |\Delta u|^2 \, dx
\end{aligned}$$

If $\beta > 2$, we let $R \to 0$ and if $\beta < 2$ we let $R \to \infty$. In both cases we obtain

$$\exists C < \infty \quad \forall u \in C^2(\mathbb{R}^n): \quad \int_{B_1} |\nabla u|^2 \, dx \leq C \int_{B_2} |u|^2 \, dy$$

which is clearly false: take $u(x) = \sin(k x_1)$ for sufficiently large $k \in \mathbb{N}$. Hence, $\beta = 2$ is the only possible choice.

10.4. Let $\Omega \subset\subset \Omega' \subset\subset \mathbb{R}^n$ be smooth domains. Let $\lambda(\Omega)$ respectively $\lambda(\Omega')$ be the corresponding first (smallest) Dirichlet eigenvalue for $-\Delta$ in Ω respectively Ω'. Then

(a) $\lambda(\Omega) < \lambda(\Omega')$

(b) $\lambda(\Omega) \leq \lambda(\Omega')$ and equality may occur

✓ (c) $\lambda(\Omega) > \lambda(\Omega')$

(d) $\lambda(\Omega) \geq \lambda(\Omega')$ and equality may occur

(e) None of the above.

The first Dirichlet eigenvalue of Ω is given by the infimum of the Rayleigh quotient:

$$\lambda(\Omega) = \inf_{v \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega |v|^2 \, dx}$$

Suppose, $v \in H_0^1(\Omega)$. Then we may extend v by zero to $\overline{v} \in H_0^1(\Omega')$ in order to obtain a competitor for $\lambda(\Omega')$. Consequently, $\lambda(\Omega) \geq \lambda(\Omega')$.

Suppose, $\lambda(\Omega) = \lambda = \lambda(\Omega')$ for some domains $\Omega \subset\subset \Omega' \subset\subset \mathbb{R}^n$. Let $u \in H_0^1(\Omega) \setminus \{0\}$ satisfy $-\Delta u = \lambda u$ in Ω and let $\overline{u} \in H_0^1(\Omega')$ be its extension by zero. Then,

$$\frac{\int_{\Omega'} |\nabla \overline{u}|^2 \, dx}{\int_{\Omega'} |\overline{u}|^2 \, dx} = \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega |u|^2 \, dx} = \inf_{v \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega |v|^2 \, dx} = \lambda = \inf_{v \in H_0^1(\Omega')} \frac{\int_{\Omega'} |\nabla v|^2 \, dx}{\int_{\Omega'} |v|^2 \, dx}.$$

Recall from Problem 5.6 that $|\overline{u}| = \overline{u}_+ + \overline{u}_- \in H_0^1(\Omega)$ with $|\nabla |\overline{u}|| = |\nabla \overline{u}|$ almost everywhere. Hence, $|\overline{u}|$ also minimises the Rayleigh quotient. Consequently we have $-\Delta |\overline{u}| = \lambda |\overline{u}|$ in Ω' and $|\overline{u}|$ is smooth by elliptic regularity. Moreover, since $|\overline{u}|$ is supported in $\overline{\Omega} \subset\subset \Omega'$, the gradient of $|\overline{u}|$ vanishes along $\partial\Omega'$ such that

$$0 = -\int_{\Omega'} \Delta |\overline{u}| \, dx = \lambda \int_{\Omega'} |\overline{u}| \, dx.$$

Hence $\lambda = 0$ which implies (via the Rayleigh quotient) that $|\overline{u}|$ is constant. Due to the Dirichlet boundary conditions, this constant must be zero, so $\overline{u}$ and hence u vanish identically in contradiction to our assumption.

10.5. In which domain Ω is the following differential operator uniformly elliptic?

$$Lu = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big), \qquad \big(a_{ij}(x_1,x_2)\big) = \begin{pmatrix} x_1^2+x_2^2 & x_1+x_2 \\ x_1+x_2 & 1 \end{pmatrix}$$

(Here, $B_1(p) = \{x \in \mathbb{R}^2 : |x-p| < 1\}$ denotes the unit ball around $p \in \mathbb{R}^2$.)

(a) $\Omega = B_1((0,1))$

(b) $\Omega = B_1((1,0))$

(c) $\Omega = B_1((0,-1))$

(d) $\Omega = B_1((-1,0))$

√ (e) None of the above.

Since $\det(a_{ij}) = x_1^2+x_2^2-(x_1+x_2)^2 = -2x_1x_2$, one of the eigenvalues of (a_{ij}) vanishes at the center of each of the given domains.

Part II. True or false?

10.6. Let $B_1 \subset \mathbb{R}^3$ be the unit ball. Then, there exists a unique $u \in C^3(\overline{B_1}) \cap H^1_0(B_1)$ satisfying $\Delta u = -1$ in B_1.

√ (a) True.

(b) False.

The equation $-\Delta u = 1$ has a unique weak solution $u \in H^1_0(B_1)$. Since both, the right hand side and the domain are smooth, elliptic regularity yields $u \in C^\infty(\overline{B_1})$.

10.7. Let $\Omega \subset \mathbb{R}^n$ be open. Let L be uniformly elliptic and bounded in divergence form with smooth coefficients. If $u \in H^1_{\mathrm{loc}}(\Omega)$ is a weak solution of $-Lu = f$ and $f \in L^2_{\mathrm{loc}}(\Omega)$, then $u \in H^2_{\mathrm{loc}}(\Omega)$.

√ (a) True.

(b) False.

The proof of the interior regularity estimate uses only local properties.

10.8. Let $\Omega \subset \mathbb{R}^n$ be open and bounded of class C^∞. There exists $m \in \mathbb{N}$ such that the embedding $W^{k,p}(\Omega) \hookrightarrow C^m(\Omega)$ does *not* hold for any $k, p \in \mathbb{N}$.

(a) True.

√ (b) False.

Let $m \in \mathbb{N}$ be arbitrary. If $k - \frac{n}{p} > m$, then the embedding $W^{k,p}(\Omega) \hookrightarrow C^m(\Omega)$ holds.

10.9. Let $\Omega \subset \mathbb{R}^n$ be open and bounded of class C^∞. For sufficiently large $k, p \in \mathbb{N}$ the embedding $W^{k,p}(\Omega) \hookrightarrow C^\ell$ holds for any $\ell \in \mathbb{N}$.

(a) True.

√ (b) False.

$W^{k,p}(\Omega)$ does not embed into $C^k(\Omega)$.

10.10. The three smallest Dirichlet eigenvalues $0 < \lambda_1, \lambda_2, \lambda_3$ of the Laplacian $-\Delta$ on the rectangle $D =]0,\pi[\times]0,3\pi[\subset \mathbb{R}^2$ are

$$\lambda_1 = 1, \qquad \lambda_2 = \tfrac{13}{9}, \qquad \lambda_3 = 2.$$

(a) True.

√ (b) False.

By elliptic regularity, weak solutions $u \in H^1_0(D)$ of $-\Delta u = \lambda u$ are smooth. (The rectangle D is not of class C^1 but in this case, one can argue that by iterated reflection in both x- and y-directions one gets a smooth bi-periodic solution on the plane.)

First we solve the corresponding one-dimensional boundary value problem:

$$\begin{cases} -\varphi'' = \mu\varphi & \text{in }]0,\pi[, \\ \varphi(x) = 0 & \text{for } x \in \{0,\pi\}, \end{cases} \qquad \Rightarrow\ \varphi_k(x) = \sin(kx), \qquad \mu_k = k^2.$$

Since $(\varphi_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(]0,\pi[)$, we have

$$u(x,y_0) = \sum_{j=1}^{\infty} \psi_j(y_0)\varphi_j(x)$$

at any fixed $y_0 \in]0,3\pi[$. Formally we compute

$$\sum_{j=1}^{\infty} \lambda\psi_j\varphi_j = \lambda u = -\Delta u = \sum_{j=1}^{\infty}(-\psi_j''\varphi_j - \psi_j\varphi_j'') = \sum_{j=1}^{\infty}(-\psi_j''\varphi_j + \mu_j\psi_j\varphi_j)$$

multiply by φ_k and integrate in x to arrive at the boundary value problem

$$\begin{cases} -\psi_k'' = (\lambda-\mu_k)\psi_k & \text{in }]0,3\pi[, \\ \psi_k(y) = 0 & \text{for } y \in \{0,3\pi\}, \end{cases} \Rightarrow\ \psi_{kn}(y) = \sin(\tfrac{ny}{3}), \quad (\lambda - \mu_k) = \frac{n^2}{9}.$$

This formal computation suggests that the eigenvectors and eigenvalues of $-\Delta$ are

$$u_{kn}(x,y) = \sin(kx)\sin(\tfrac{1}{3}ny), \qquad \lambda_{kn} = k^2 + \frac{n^2}{9}, \qquad n,k \in \mathbb{N}$$

and that they are the only ones, namely that if $-\Delta u = \lambda u$, then u is a linear combination of the finitely many u_{kn} with $\lambda_{kn} = \lambda$. Indeed, assume that u is L^2-orthogonal to them and still satisfies $-\Delta u = \lambda u$. Then u is also L^2-orthogonal to every other u_{kn} as eigenvectors for different eigenvalues are orthogonal. Being orthogonal to all u_{kn} implies $u = 0$ as $\{u_{kn} : n,k \in \mathbb{N}\}$ is a Hilbert basis of $L^2(D)$.

Since $k,n \geq 1$, the first three eigenvalues are

$$\lambda_{11} = \frac{10}{9}, \qquad \lambda_{12} = \frac{13}{9}, \qquad \lambda_{13} = 2.$$

10.11. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $f \in L^2(\Omega)$. Let $u \in H^1(\Omega)$ be a weak solution of $-\Delta u = f$. Then, for any $\Omega' \subset\subset \Omega$, there exists a constant $C < \infty$ which depends only on the pair Ω, Ω' (but not on u nor k) such that

$$\forall k \in \mathbb{R}: \quad \int_{\Omega'} |\nabla u|^2 \, dx \leq C\Big(\int_\Omega |u-k|^2 \, dx + \int_\Omega |f|^2 \, dx\Big).$$

√ (a) True.

(b) False.

Given $\Omega' \subset\subset \Omega$, let $\varphi \in C_c^\infty(\Omega)$ be a cut-off function satisfying $0 \leq \varphi \leq 1$ and $\varphi|_{\Omega'} \equiv 1$. Since $u \in H^1(\Omega)$ is a weak solution of $-\Delta u = f$, we obtain

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$$

for every $v \in H_0^1(\Omega)$, in particular for $v = (u-k)\varphi^2$. With this choice,

$$\nabla u \cdot \nabla v = |\nabla u|^2 \varphi^2 + 2(u-k)\varphi \nabla u \cdot \nabla \varphi.$$

Hence, applying Cauchy–Schwarz and Young's inequality

$$\begin{aligned}
\int_\Omega |\nabla u|^2 \varphi^2 \, dx &= \int_\Omega (u-k)\varphi^2 f \, dx - \int_\Omega 2(u-k)\varphi \nabla u \cdot \nabla\varphi \, dx \\
&\leq \Big(\int_\Omega |u-k|^2 \, dx\Big)^{\frac12} \Big(\int_\Omega |f|^2 \, dx\Big)^{\frac12} + \Big(\int_\Omega 4|u-k|^2 |\nabla\varphi|^2 \, dx\Big)^{\frac12} \Big(\int_\Omega |\nabla u|^2 \varphi^2 \, dx\Big)^{\frac12} \\
&\leq \big(\tfrac12 + \|\nabla\varphi\|_{C^0}\big)\Big(\int_\Omega |u-k|^2 \, dx\Big) + \frac12 \Big(\int_\Omega |f|^2 \, dx\Big) + \frac12 \Big(\int_\Omega |\nabla u|^2 \varphi^2 \, dx\Big).
\end{aligned}$$

Hence, by absorbing the gradient term,

$$\int_{\Omega'} |\nabla u|^2 \, dx \leq \int_\Omega |\nabla u|^2 \varphi^2 \, dx \leq C \int_\Omega |u-k|^2 \, dx + \int_\Omega |f|^2 \, dx,$$

where the constant $C = \Big(1 + 2\|\nabla\varphi\|_{C^0}\Big)$ depends on the pair Ω, Ω' but not on u or k.

Remark. The statement is known as *Caccioppoli inequality.*

10.12. Let $\Omega \subset \mathbb{R}^2$ be open and bounded. Provided we have a solution $u \in C^2(\overline{\Omega})$ of the equation

$$(1+u_1^2)\,u_{22} - 2u_1u_2\,u_{12} + (1+u_2^2)\,u_{11} = 0 \quad \text{ in } \Omega,$$

where the subscripts denote partial derivatives, then this equation can be rewritten in divergence form.

√ (a) True.

(b) False.

The equation is of the form $-Lu = 0$, where

$$Lu = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big), \qquad (a_{ij}) = \begin{pmatrix} (1+|\nabla u|^2)^{-\frac{1}{2}} & 0 \\ 0 & (1+|\nabla u|^2)^{-\frac{1}{2}} \end{pmatrix}.$$

Indeed, computing

$$\frac{\partial}{\partial x_i}\Big(1+|\nabla u|^2\Big)^{-\frac{1}{2}} = -\frac{u_1u_{1i}+u_2u_{2i}}{(1+|\nabla u|^2)^{\frac{3}{2}}}$$

we can read off that the coefficient of u_{11} is

$$-\frac{u_1^2}{(1+|\nabla u|^2)^{\frac{3}{2}}} + \frac{1}{(1+|\nabla u|^2)^{\frac{1}{2}}} = \frac{1+u_2^2}{(1+|\nabla u|^2)^{\frac{3}{2}}}.$$

Analogously, we can read off the coefficients of u_{22} and u_{12} and see that after multiplication with $(1+|\nabla u|^2)^{\frac{3}{2}}$ we obtain the given equation.

Since $u \in C^2(\overline{\Omega})$ implies that $(1+|\nabla u|^2)^{-\frac{1}{2}}$ is in $C^1(\Omega)$ and bounded from above and below, we obtain that L is uniformly elliptic and bounded in divergence form with C^1 coefficients.

10.13. Let $B_1 \subset \mathbb{R}^2$ be the unit disc. Then, the problem

$$\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) = 1, \qquad \Big(a_{ij}(x_1,x_2)\Big) = \begin{pmatrix} 2 & \frac{x_1x_2}{|x_1x_2|} \\ \frac{x_1x_2}{|x_1x_2|} & 2 \end{pmatrix}$$

has a weak solution $u \in H^1_0(B_1)$.

√ (a) True.

(b) False.

The matrix (a_{ij}) is symmetric with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$. Hence, ellipticity holds and

$$(u,v)_a := \int_{B_1} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx$$

defines a scalar product which is equivalent to the standard scalar product on $(\cdot,\cdot)_{H^1_0(B_1)}$. The claim follows from the Riesz representation theorem.

10.14. Assume $u \in H^1(\Omega)$ is harmonic, namely solves $-\Delta u = 0$ (weakly and thus classically) on a bounded smooth domain Ω. If $g := u|_{\partial\Omega} \in C^0(\partial\Omega)$, then $u \in C^0(\overline{\Omega})$.

√ (a) True.

(b) False.

We claim that any weak solution $v \in H^1(\Omega)$ of $-\Delta v = 0$ with $v|_{\partial\Omega} = g$ satisfies

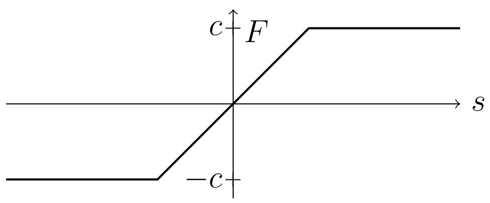
$$\|v\|_{L^\infty(\Omega)} \leq \|v|_{\partial\Omega}\|_{L^\infty(\partial\Omega)}. \tag{$*$}$$

In fact, by Lemma 9.3.2, v is the unique minimiser of the Dirichlet energy

$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx$$

among all functions $H^1(\Omega)$ with the same trace g. Now consider $c := \|g\|_{L^\infty(\partial\Omega)}$ and

$$F(s) = \begin{cases} c & \text{if } s > c, \\ s & \text{if } -c \leq s \leq c, \\ -c & \text{if } s < -c. \end{cases}$$



Then, $F \circ v \in H^1(\Omega)$ with the same trace g and $E(F \circ v) \leq E(v)$. By uniqueness of the minimiser, $F \circ v = v$. Therefore $|v| \leq c$ which proves the claim.

Let u be harmonic in Ω and let $g = u|_{\partial\Omega} \in C^0(\partial\Omega)$. Let $(g_k)_{k\in\mathbb{N}}$ be a sequence in $C^\infty(\Omega)$ such that $g_k|_{\partial\Omega} \to g$ in $C^0(\partial\Omega)$ as $k \to \infty$. Let $v_k \in H^1_0(\Omega)$ be the weak solution of $-\Delta v_k = f_k$ where $f_k := \Delta g_k \in C^\infty(\Omega)$. By elliptic regularity, $v_k \in C^\infty(\Omega)$ and $v_k|_{\partial\Omega} = 0$. Thus, $u_k := v_k + g_k \in C^\infty(\Omega)$ satisfies $\Delta u_k = 0$ and $u_k|_{\partial\Omega} = g_k|_{\partial\Omega}$. Moreover, by $(*)$ $\|u_k - u\|_{L^\infty(\Omega)} \leq \|g_k - g\|_{L^\infty(\partial\Omega)} \to 0$ as $k \to \infty$. As uniform limit of continuous functions, u is continuous in $\overline{\Omega}$.

10.15. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $V \in C^\infty(\overline{\Omega})$. Recall that the operator $A = -\Delta + V\colon D_A \subset L^2(\Omega) \to L^2(\Omega)$, where $D_A = \{u \in C^2(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ is closable. The domain of its closure $\overline{A}$ is $D_{\overline{A}} = H^2(\Omega) \cap H^1_0(\Omega)$.

√ (a) True.

(b) False.

"$\subseteq$" Let $u \in D_{\overline{A}}$. By definition, there exists a sequence $(u_k)_{k\in\mathbb{N}}$ in D_A such that

$$(u_k, Au_k) \xrightarrow{k\to\infty} (u, f) \quad \text{ in } L^2(\Omega) \times L^2(\Omega)$$

for some $f \in L^2(\Omega)$. By elliptic regularity,

$$\begin{aligned}\|u_k - u_\ell\|_{H^2(\Omega)} &\leq C\|\Delta(u_k - u_\ell)\|_{L^2(\Omega)} = C\|{-A}(u_k - u_\ell) + V(u_k - u_\ell)\|_{L^2(\Omega)}\\ &\leq C\|Au_\ell - Au_k\|_{L^2(\Omega)} + C\|V\|_{C^0(\Omega)}\|u_k - u_\ell\|_{L^2(\Omega)}.\end{aligned}$$

Hence, $(u_k)_{k\in\mathbb{N}}$ is Cauchy in $H^2(\Omega)$ and therefore convergent in $H^2(\Omega)$. Since $(u_k)_{k\in\mathbb{N}}$ also converges in $L^2(\Omega)$, the respective limits must coincide: therefore, $u \in H^2(\Omega)$. Moreover, convergence in $H^2(\Omega)$ implies convergence in $H^1(\Omega)$ and $H^1_0(\Omega)$ is a closed subspace of $H^1(\Omega)$. Therefore, $u_k \in D_A \subset H^1_0(\Omega)$ implies $u \in H^1_0(\Omega)$. To conclude, $u \in H^2(\Omega) \cap H^1_0(\Omega)$.

"$\supseteq$" Let $u \in H^2(\Omega) \cap H^1_0(\Omega)$. Then, $-\Delta u \in L^2(\Omega)$ and there exists a sequence $(f_k)_{k\in\mathbb{N}}$ in $C^\infty_c(\Omega)$ such that $f_k \to -\Delta u$ in $L^2(\Omega)$. For each $k \in \mathbb{N}$, let $u_k \in H^1_0(\Omega)$ be the weak solution of $-\Delta u_k = f_k$ in Ω. By elliptic regularity, $u_k \in C^\infty(\overline{\Omega})$. In particular, $u_k \in C^2(\overline{\Omega}) \cap H^1_0(\Omega) = D_A$. Moreover,

$$\begin{aligned}\|u_k - u\|_{L^2(\Omega)} &\leq \|u_k - u\|_{H^2(\Omega)} \leq C\|f_k - (-\Delta u)\|_{L^2(\Omega)} \xrightarrow{k\to\infty} 0,\\ \|Au_k - Au_\ell\|_{L^2(\Omega)} &\leq \|f_k - f_\ell\|_{L^2(\Omega)} + \|V\|_{C^0(\Omega)}\|u_k - u_\ell\|_{L^2(\Omega)}.\end{aligned}$$

Hence, $D_A \ni u_k \to u$ in $L^2(\Omega)$ and $(Au_k)_{k\in\mathbb{N}}$ is Cauchy in $L^2(\Omega)$ and thus convergent. Consequently $u \in D_{\overline{A}}$.