11.1. Integration by parts

Let $w \in H^1_0(B_r)$. By definition of H^1_0, there exists a sequence $(w_k)_{k\in\mathbb{N}}$ in $C^\infty_c(B_r)$ such that $\|w - w_k\|_{H^1(B_r)} \to 0$ as $k \to \infty$. Moreover, integration by parts yields

$$\int_{B_r} \frac{\partial w_k}{\partial x_i}\,dx = \int_{\partial B_r} w_k\,d\sigma = 0.$$

Therefore,

$$\begin{aligned}\Big|\int_{B_r} \frac{\partial w}{\partial x_i}\,dx\Big| &= \Big|\int_{B_r} \frac{\partial w}{\partial x_i} - \frac{\partial w_k}{\partial x_i}\,dx\Big| \\ &\leq \int_{B_r} |\nabla w - \nabla w_k|\,dx \leq |B_r|^{\frac12}\|\nabla w - \nabla w_k\|_{L^2(B_r)} \xrightarrow{k\to\infty} 0.\end{aligned}$$

Since the left hand side does not depend on k it must vanish as claimed.

11.2. Linear transformation

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be linear, symmetric and invertible. Given $v \in H^1(B_r^+)$, let $w = v \circ T^{-1}$. Then, using symmetry of T,

$$\begin{aligned}\frac{\partial w}{\partial \nu} &= \nu\cdot\nabla w = (Te_n)\cdot\nabla(v\circ T^{-1}) = (Te_n)\cdot\Big(T^{-1}((\nabla v)\circ T^{-1})\Big) \\ &= e_n\cdot\Big(TT^{-1}((\nabla v)\circ T^{-1})\Big) = e_n\cdot((\nabla v)\circ T^{-1}) = \frac{\partial v}{\partial x_n}\circ T^{-1}.\end{aligned}$$

With the change of variables $x = T^{-1}y$ we obtain

$$\begin{aligned}\int_{B_r^+}\Big|\frac{\partial v}{\partial x_n}(x)\Big|^2\,dx &= \int_{T(B_r^+)}\Big|\frac{\partial v}{\partial x_n}(T^{-1}y)\Big|^2|\det(T^{-1})|\,dy \\ &= |\det(T^{-1})|\int_{T(B_r^+)}\Big|\frac{\partial w}{\partial \nu}(y)\Big|^2\,dy.\end{aligned}$$

11.3. Basic iteration lemma

Let $f\colon\,]0,R_0] \to [0,\infty[$ be a non-decreasing function satisfying

$$\forall \rho \in\,]0,R_0]: \quad f\Big(\frac{\rho}{2}\Big) \leq \Big(\frac12\Big)^\alpha f(\rho)$$

for some $\alpha > 0$. Given $0 < r < R \leq R_0$, let $N \in \mathbb{N}$ such that $2^{-N-1}R < r \leq 2^{-N}R$. By monotonicity of f and iteration of the hypothesis, we obtain

$$f(r) \leq f\Big(\frac{R}{2^N}\Big) \leq \Big(\frac12\Big)^{\alpha N} f(R) = \Big(\frac{2}{R}\Big)^\alpha\Big(\frac{R}{2^{N+1}}\Big)^\alpha f(R) < \Big(\frac{2r}{R}\Big)^\alpha f(R).$$

11.4. Technical iteration lemma

Let $f\colon]0,R_0]\to[0,\infty[$ be a non-decreasing function satisfying for some coefficients $A,B\geq 0$, some exponents $0<\beta<\alpha$ and some $\varepsilon\geq 0$ the following inequality.

$$\forall 0<r<R\leq R_0:\qquad f(r)\leq A\Bigl(\Bigl(\frac{r}{R}\Bigr)^{\alpha}+\varepsilon\Bigr)f(R)+BR^{\beta}.$$

(a) For any $0<R\leq R_0$ and any $\tau\in\,]0,1[$ we have by assumption

$$f(\tau R)\leq A(\tau^{\alpha}+\varepsilon)f(R)+BR^{\beta}.$$

Since increasing A weakens the assumption, we may assume $A>\frac{1}{2}$ without loss of generality. Suppose, $\varepsilon\leq(2A)^{-\frac{\alpha}{\alpha-\gamma}}=:\varepsilon_0$ for some $\gamma\in\,]\beta,\alpha[$. Note that since $\alpha>\gamma$ and $2A>1$, there exists $\tau\in\,]0,1[$ such that $2A\tau^{\alpha}=\tau^{\gamma}$. In particular, $\varepsilon\leq(2A)^{-\frac{\alpha}{\alpha-\gamma}}=\tau^{\alpha}$ and $A(\tau^{\alpha}+\varepsilon)\leq 2A\tau^{\alpha}=\tau^{\gamma}$. Therefore, $f(\tau R)\leq\tau^{\gamma}f(R)+BR^{\beta}$ for any $0<R\leq R_0$ as claimed.

(b) In part (a), we proved the claim for $k=1$. Suppose, the inequality

$$f(\tau^k R)\leq\tau^{k\gamma}f(R)+BR^{\beta}\tau^{(k-1)\beta}\sum_{n=0}^{k-1}\tau^{n(\gamma-\beta)}$$

is true for some $k\in\mathbb{N}$. Then, by (a)

$$\begin{aligned}
f(\tau^{k+1}R)&\leq\tau^{\gamma}f(\tau^k R)+B(\tau^k R)^{\beta}\\
&\leq\tau^{(k+1)\gamma}f(R)+\tau^{\gamma}BR^{\beta}\tau^{(k-1)\beta}\sum_{n=0}^{k-1}\tau^{n(\gamma-\beta)}+B(\tau^k R)^{\beta}\\
&=\tau^{(k+1)\gamma}f(R)+BR^{\beta}\tau^{k\beta}\Bigl(\tau^{\gamma-\beta}\sum_{n=0}^{k-1}\tau^{n(\gamma-\beta)}+1\Bigr)\\
&=\tau^{(k+1)\gamma}f(R)+BR^{\beta}\tau^{k\beta}\sum_{n=0}^{k}\tau^{n(\gamma-\beta)}
\end{aligned}$$

and the claim follows by induction.

(c) Given $0<r<R\leq R_0$ let $N\in\mathbb{N}_0$ such that $\tau^{N+1}R<r\leq\tau^N R$. Then, by (b)

$$\begin{aligned}
f(r)\leq f(\tau^N R)&\leq\tau^{N\gamma}f(R)+BR^{\beta}\tau^{(N-1)\beta}\sum_{n=0}^{N-1}\tau^{n(\gamma-\beta)}\\
&\leq\tau^{-\gamma}\tau^{(N+1)\gamma}f(R)+BR^{\beta}\tau^{(N+1)\beta}\frac{\tau^{-2\beta}}{1-\tau^{(\gamma-\beta)}}\\
&\leq\tau^{-\gamma}\Bigl(\frac{r}{R}\Bigr)^{\gamma}f(R)+Br^{\beta}\frac{\tau^{-2\beta}}{1-\tau^{(\gamma-\beta)}}\\
&\leq C\Bigl(\Bigl(\frac{r}{R}\Bigr)^{\beta}f(R)+Br^{\beta}\Bigr),
\end{aligned}$$

where $C:=\max\Bigl\{\tau^{-\gamma},\frac{\tau^{-2\beta}}{1-\tau^{(\gamma-\beta)}}\Bigr\}$ depends only on α,β,γ and A.

11.5. Interpolation inequality

Towards a contradiction, suppose there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in X and some $\varepsilon_0 > 0$ such that

$$\forall k \in \mathbb{N}: \quad 1 = \|x_k\|_Y \geq \varepsilon_0 \|x_k\|_X + k\|x_k\|_Z.$$

Then, $\|x_k\|_X \leq \frac{1}{\varepsilon_0}$ and $\|x_k\|_Z \leq \frac{1}{k}$ for every $k \in \mathbb{N}$. Thus, the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in X and since the embedding $X \hookrightarrow Y$ is compact, there exists a subsequence $(x_k)_{k\in\Lambda\subset\mathbb{N}}$ and some $y \in Y$ such that $\|x_k - y\|_Y \to 0$ as $\Lambda \ni k \to \infty$. Since the embedding $Y \hookrightarrow Z$ is continuous, we also have $\|x_k - y\|_Z \to 0$ as $\Lambda \ni k \to \infty$. Consequently,

$$1 = \lim_{\Lambda\ni k\to\infty} \|x_k\|_Y = \|y\|_Y, \qquad\qquad \|y\|_Z = \lim_{\Lambda\ni k\to\infty} \|x_k\|_Z = 0$$

which is a contradiction.

11.6. Abstract method of continuity

Given $A_0, A_1 \in L(X,Y)$ let $A_t = (1-t)A_0 + tA_1$ for every $t \in [0,1]$ and assume that

$$\exists C < \infty \quad \forall t \in [0,1] \quad \forall x \in X: \quad \|x\|_X \leq C\|A_t x\|_Y. \tag{$*$}$$

The claim is equivalence of the statements

(i) A_0 is surjective.

(ii) A_1^* is injective with closed image.

(a) Let $I := \{t \in [0,1] : A_t \text{ is surjective}\}$. If we assume statement (i), then $0 \in I$. If we assume statement (ii), then $1 \in I$ by Satz 6.2.2. Therefore, $I \neq \emptyset$ in both cases.

(b) Let $t_0 \in I := \{t \in [0,1] : A_t \text{ is surjective}\}$. Assumption $(*)$ implies that A_{t_0} is also injective and that the inverse is continuous: $A_{t_0}^{-1} \in L(Y,X)$. For any $t \in [0,1]$, we have

$$\begin{aligned} A_t &= A_{t_0} - (A_{t_0} - A_t) = \Big(1 - (A_{t_0} - A_t)A_{t_0}^{-1}\Big)A_{t_0}, \\ A_{t_0} - A_t &= (1-t_0)A_0 + t_0A_1 - (1-t)A_0 - tA_1 = (t-t_0)(A_0 - A_1). \end{aligned}$$

Let $B := (A_{t_0} - A_t)A_{t_0}^{-1} \in L(Y,Y)$. By Satz 2.2.7 the operator $(1-B)$ is invertible with inverse $(1-B)^{-1} \in L(Y,Y)$ and in particular surjective, if $\|B\| < 1$. Since

$$\|B\| \leq \|A_{t_0} - A_t\|\|A_{t_0}^{-1}\| = |t-t_0|\|A_0 - A_1\|\|A_{t_0}^{-1}\|$$

we guarantee surjectivity of $(1-B)$ if $t \in [0,1]$ satisfies $|t-t_0| < (\|A_0 - A_1\|\|A_{t_0}^{-1}\|)^{-1}$. In this case we obtain that A_t is surjective, since A_{t_0} is surjective by assumption. Therefore, the set $I \subset [0,1]$ is open.

(c) Let $(t_k)_{k\in\mathbb{N}}$ be a sequence in I such that $t_k \to t_\infty$ as $k \to \infty$ for some $t_\infty \in [0,1]$. We claim that $A_{t_\infty} \in L(X,Y)$ is surjective. Let $y \in Y$ be arbitrary. Since $t_k \in I$, there exists $x_k \in X$ such that $A_{t_k}x_k = y$ for every $k \in \mathbb{N}$. Moreover, by assumption $(*)$,

$$\begin{aligned}
\|x_k - x_n\|_X &\leq C\|A_{t_k}(x_k - x_n)\|_Y \\
&= C\|A_{t_k}x_k - A_{t_n}x_n + (A_{t_n} - A_{t_k})x_n\|_Y \\
&= C\|(A_{t_n} - A_{t_k})x_n\|_Y \\
&\leq C\|A_{t_n} - A_{t_k}\|\|x_n\|_X \\
&\leq C^2|t_k - t_n|\|A_0 - A_1\|\|A_{t_n}x_n\|_Y = C^2|t_k - t_n|\|A_0 - A_1\|\|y\|_Y
\end{aligned}$$

which implies that $(x_k)_{k\in\mathbb{N}}$ is a Cauchy-sequence in X. Since $(X, \|\cdot\|_X)$ is complete, $(x_k)_{k\in\mathbb{N}}$ has a limit $x_\infty \in X$. Moreover,

$$\begin{aligned}
\|y - A_{t_\infty}x_\infty\|_Y &= \|A_{t_k}x_k - A_{t_\infty}x_\infty\|_Y \\
&= \|(A_{t_k} - A_{t_\infty})x_k + A_{t_\infty}(x_k - x_\infty)\|_Y \\
&\leq C\|A_{t_k} - A_{t_\infty}\|\|y\|_Y + \|A_{t_\infty}\|\|x_k - x_\infty\|_X \\
&\leq C|t_\infty - t_k|\|A_0 - A_1\|\|y\|_Y + \|A_{t_\infty}\|\|x_k - x_\infty\|_X \xrightarrow{k\to\infty} 0.
\end{aligned}$$

Hence, $A_{t_\infty}x_\infty = y$. Since $y \in Y$ is arbitrary, $t_\infty \in I$ follows. Therefore, the set $I \subset [0,1]$ is closed.

Since $[0,1]$ is a connected topological space and $I \subset [0,1]$ both open and closed by (b) and (c), we have either $I = \emptyset$ or $I = [0,1]$. According to Satz 6.2.2, A_1 is surjective if and only if A_1^* is injective with closed image. Hence, equivalence of (i) and (ii) follows:

$$\begin{array}{ccccccccc}
\text{(i)} & \Leftrightarrow & 0 \in I & \Rightarrow & I = [0,1] & \Rightarrow & A_1 \text{ surjective} & \Leftrightarrow & \text{(ii)} \\
\text{(ii)} & \Leftrightarrow & 1 \in I & \Rightarrow & I = [0,1] & \Rightarrow & A_0 \text{ surjective} & \Leftrightarrow & \text{(i)}
\end{array}$$