Part I. Multiple choice questions

12.1. Let $Q =]0,1[^n \subset \mathbb{R}^n$ and let $\Gamma = \{0,1\}^n \subset \mathbb{R}^n$ be the set of vertices of Q. Let $u \in H_0^1(Q)$ be a weak solution of $-\Delta u = \lambda u$ in Q for some $\lambda \in \mathbb{R}$. Then,

(a) $u \in C^\infty(Q)$ but $u \notin C^\infty(\overline{Q})$

(b) $u \in C^\infty(\overline{Q} \setminus \Gamma)$ but $u \notin C^\infty(\overline{Q})$

√ (c) $u \in C^\infty(\overline{Q})$

(d) The answer depends on the dimension $n \in \mathbb{N}$.

(e) None of the above.

Since the domain Q is not of class C^1, we can not apply the boundary regularity theory. However, we can do the following reflection trick. Let $\{e_1, \ldots, e_n\}$ be the standard basis in $\mathbb{R}^n$. Let $Q_0 = Q$ and $u_0 = 0$. For every $k \in \{1, \ldots, n\}$ we define the connected, open domain $Q_k \subset \mathbb{R}^n$ to be the interior of the closure of

$$Q_{k-1} \cup (Q_{k-1} + e_k) \cup (Q_{k-1} - e_k)$$

and u_k to be the odd reflection of u_{k-1} in both, direction e_k and $-e_k$ along the corresponding faces of Q_{k-1}. Provided that $u_{k-1} \in H_0^1(Q_{k-1})$ is a weak solution of the equation $-\Delta u = \lambda u$ in Q_{k-1} we have that $u_k \in H_0^1(Q_k)$ is a weak solution of the equation $-\Delta u = \lambda u$ in Q_k. The proof of this fact is analogous to the solution of problem 9.2. By interior elliptic regularity, u_k is smooth in the interior of Q_k. Iterating this argument for all $k \in \{1, \ldots, n\}$ proves $u \in C^\infty(\overline{Q})$ because $Q_n =]-1,2[^n \supset\supset Q$.

Remark. The argument above shows that if $\lambda \in \mathbb{R}$ is a Dirichlet eigenvalue of $-\Delta$ in Q, then λ is also a Dirichlet eigenvalue of $-\Delta$ in the larger domain $Q_n \supset\supset Q$. However, even if λ is the first eigenvalue of $-\Delta$ in Q, it is *not* the first eigenvalue of $-\Delta$ in Q_n since the corresponding eigenfunction u_n changes sign.

12.2. Let $\Omega \subset \mathbb{R}^n$ be a smooth and connected open domain. Suppose $u \in C^2(\overline{\Omega})$ satisfies $u \geq 0$, $u \not\equiv 0$, $u|_{\partial\Omega} = 0$ and $\Delta u + \lambda u = 0$ for some $\lambda \in \mathbb{R}$. Then,

(a) λ is the largest negative Dirichlet eigenvalue of the operator $-\Delta$.

(b) $\lambda = 0$.

√ (c) λ is the smallest positive Dirichlet eigenvalue of the operator $-\Delta$.

(d) λ is a positive eigenvalue of $-\Delta$, but not necessarily the smallest one.

(e) None of the above.

Since $u \not\equiv 0$ and $u|_{\partial\Omega} = 0$ integration by parts yields

$$0 < \int_\Omega |\nabla u|^2 \, dx = -\int_\Omega u \Delta u \, dx = \lambda \int_\Omega u^2 \, dx \qquad \Rightarrow \; \lambda > 0.$$

We claim that λ equals the smallest Dirichlet eigenvalue λ_1 of $-\Delta$. Indeed, recall that λ_1 has a unique eigenfunction u_1 (up to multiples), which is smooth up to the boundary by elliptic regularity. Recall also that $u_1 > 0$ in Ω, up to changing its sign. Consider now the biggest constant $\mu \geq 0$ such that $u - \mu u_1 \geq 0$ everywhere on $\overline{\Omega}$. The function $v := u - \mu u_1 \geq 0$ vanishes at the boundary and satisfies

$$-\Delta v = \lambda u - \lambda_1 \mu u_1 \geq \lambda_1 u - \lambda_1 \mu u_1 = \lambda_1 v \geq 0$$

since $\lambda \geq \lambda_1$. Hence, by the strong maximum principle, either $v \equiv 0$, in which case we have that u is a multiple of u_1 and so $\lambda = \lambda_1$, or $v > 0$ on Ω. Assuming we are in the second case, we want to reach a contradiction. By E. Hopf's lemma (applied to the function $-v$, with $a_{ij} := \delta_{ij}$ and $c := 0$), we have $\partial_\nu v < 0$. Choose now any $\eta > 0$ so small that $\partial_\nu(v - \eta u_1) < 0$ on $\partial\Omega$. By smoothness of Ω, if $\varepsilon > 0$ is small enough then the set $C_\varepsilon := \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) \leq \varepsilon\}$ can be expressed as

$$C_\varepsilon = \{y - t\nu(y) : y \in \partial\Omega, \; 0 \leq t \leq \varepsilon\}.$$

By uniform continuity of ∇u and ∇v, if ε is small enough we also have

$$\partial_{\nu(y)}(v - \eta u_1)(y - t\nu(y)) < 0$$

for all $y \in \partial\Omega$ and all $0 \leq t \leq \varepsilon$, hence

$$(v - \eta u_1)(y - t\nu(y)) = -\int_0^t \partial_{\nu(y)} u(y - s\nu(y)) \, ds \geq 0$$

and we get $u \geq (\mu + \eta) u_1$ on C_ε. On the other hand, $\Omega \setminus C_\varepsilon \subset\subset \Omega$, so here v is bounded from below by a positive constant and we can find $\eta' > 0$ such that

$$v - \eta' u_1 \geq 0 \quad \text{ in } \Omega \setminus C_\varepsilon.$$

So $u \geq (\mu + \min\{\eta, \eta'\}) u_1$ on $\overline{\Omega}$, contradicting the maximality of μ.

12.3. For which $n \in \mathbb{N}$ is the following statement true? Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain of class C^1. Then,

$$\forall \varepsilon > 0 \quad \exists C < \infty \quad \forall u \in H^1(\Omega): \quad \|u\|_{L^4(\Omega)} \leq \varepsilon \|\nabla u\|_{L^2(\Omega)} + C\|u\|_{L^2(\Omega)}$$

(a) for all $n \in \mathbb{N}$.

(b) only for $n \in \{1, 2, 3, 4\}$.

√ (c) only for $n \in \{1, 2, 3\}$.

(d) only for $n \in \{2, 3\}$.

(e) None of the above.

As Ω is bounded, the embedding $L^4(\Omega) \hookrightarrow L^2(\Omega)$ is continuous by Hölder's inequality.

If $n = 1$ the embedding $H^1(\Omega) \hookrightarrow C^0(\Omega)$ is compact and $C^0(\Omega) \hookrightarrow L^4(\Omega)$ continuously.

If $n = 2$ the embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact for any $1 \leq q < \infty$.

If $n = 3$ the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ is compact because $4 < \frac{2n}{n-2} = 6$.

The right hand side of the inequality is $\varepsilon\|\nabla u\|_{L^2} + C\|u\|_{L^2} = \varepsilon\|u\|_{H^1} + (C - \varepsilon)\|u\|_{L^2}$. Hence, for $n \in \{1, 2, 3\}$, the claim follows from the interpolation inequality proven in problem 11.5 (respectively Lemma 10.4.2) with $X = H^1$, $Y = L^4$ and $Z = L^2$.

If $n = 4$ the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ is only continuous and not compact. In fact, we can show that the statement does not hold in this case: Let $\Omega = B_1(0) \subset \mathbb{R}^4$. Given any $u \in C_c^\infty(\Omega)$ and any $k \in \mathbb{N}$, let

$$u_k(y) := k\, u(ky), \qquad \Rightarrow \; \nabla u_k(y) = k^2 \nabla u(ky)$$

which we extend by zero to functions in Ω. Recall the scaling properties

$$\begin{aligned}
\int_\Omega |u(x)|^2\, dx &= \int_\Omega |u(ky)|^2 k^4\, dy = k^2 \int_\Omega |u_k(y)|^2\, dy, \\
\int_\Omega |u(x)|^4\, dx &= \int_\Omega |u(ky)|^4 k^4\, dy = \int_\Omega |u_k(y)|^4\, dy, \\
\int_\Omega |(\nabla u)(x)|^2\, dx &= \int_\Omega |(\nabla u)(ky)|^2 k^4\, dy = \int_\Omega |(\nabla u_k)(y)|^2\, dy.
\end{aligned}$$

Assuming the statement to be true, we obtain

$$\|u\|_{L^4(\Omega)} = \|u_k\|_{L^4(\Omega)} \leq \varepsilon\|\nabla u_k\|_{L^2(\Omega)} + C\|u_k\|_{L^2(\Omega)} = \varepsilon\|\nabla u\|_{L^2(\Omega)} + \frac{C}{k}\|u\|_{L^2(\Omega)}.$$

Letting $k \to \infty$ yields $\|u\|_{L^4} \leq \varepsilon\|\nabla u\|_{L^2}$ which can not be true for arbitrary $\varepsilon > 0$ unless $u \equiv 0$.

12.4. Assume $-\Delta u = f$ in the open unit ball $B_1 \subset \mathbb{R}^n$ for some function $f \in C^0(\overline{B_1})$ satisfying $|f(x)| \leq 1$ for all $x \in \overline{B_1}$ and $|f| \not\equiv 1$ and some $u \in C^2(\overline{B_1})$ vanishing along the boundary ∂B_1. Then, for every $x \in B_1$

(a) $|u(x)| < \frac{1}{2n}|x|^2$.

(b) $|u(x)| \leq \frac{1}{2n}|x|^2$ and equality may occur at some point in B_1.

√ (c) $|u(x)| < \frac{1}{2n}\left(1 - |x|^2\right)$.

(d) $|u(x)| \leq \frac{1}{2n}\left(1 - |x|^2\right)$ and equality may occur at some point in B_1.

(e) None of the above.

The example $u(x) := \frac{1}{4n}(1 - |x|^2)$ with $-\Delta u = \frac{1}{2} =: f$ shows that $\frac{1}{2n}|x|^2$ is not an upper bound for $|u(x)|$ in general.

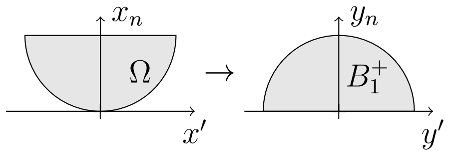
Observe that the function $v(x) := \frac{1}{2n}(1 - |x|^2)$ satisfies $-\Delta v = 1$ and $v|_{\partial B_1} = 0$. Hence $-\Delta(u - v) = f - 1 \leq 0$, so by the maximum principle

$$\max_{\overline{B_1}}(u - v) = \max_{\partial B_1}(u - v) = 0 \qquad \Rightarrow \quad u \leq v.$$

Notice that $u(x) < v(x)$ for every interior point $x \in B_1$, since otherwise the strong maximum principle would give that $u - v$ is constant, hence $u - v = 0$ by the boundary conditions. However, $-\Delta u = f \not\equiv 1 = -\Delta v$. Replacing u and f with $-u$ and $-f$ in the argument above, we obtain $-v < u < v$ in B_1.

12.5. Given $x = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$ we define $x' = (x^1, x^2, \dots, x^{n-1}) \in \mathbb{R}^{n-1}$. Consider the domain $\Omega = \{x = (x', x^n) \in \mathbb{R}^n : |(x', x^n - 1)| < 1,\ x^n < 1\}$ and let

$$\begin{aligned}\Phi \colon \Omega &\to B_1^+\\ (x', x^n) &\mapsto \Big(x', x^n - 1 + \sqrt{1 - |x'|^2}\Big)\end{aligned}$$



be a diffeomorphism flattening the lower part $\Gamma = \partial\Omega \cap \partial B_1(0,1)$ of the boundary of Ω. Let $u \in H^1(\Omega)$ be a weak solution with vanishing trace on Γ of the equation

$$-\Delta u = \operatorname{div} f$$

where $f = (f^1, \dots, f^n)^\intercal \in C^{1,\alpha}(\Omega; \mathbb{R}^n)$. Then, the equation solved by $v := u \circ \Phi^{-1}$ is

$$-\operatorname{div}(a \cdot \nabla v) = \operatorname{div}\Big(b \cdot (f \circ \Phi^{-1})\Big)$$

with $(n \times n)$-matrices $a = a(y)$ and $b = b(y)$ given by

(a) $a = \mathrm{Id}_{\mathbb{R}^n}, \qquad b = \mathrm{Id}_{\mathbb{R}^n}$

(b) $a = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & \frac{(y')^\intercal}{\sqrt{1-|y'|^2}} \\ \frac{y'}{\sqrt{1-|y'|^2}} & \frac{1+|y'|^2}{1-|y'|^2} \end{pmatrix}, \qquad b = \mathrm{Id}_{\mathbb{R}^n}$

(c) $a = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & \frac{(y')^\intercal}{\sqrt{1-|y'|^2}} \\ \frac{y'}{\sqrt{1-|y'|^2}} & \frac{1}{1-|y'|^2} \end{pmatrix}, \qquad b = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & \frac{(y')^\intercal}{\sqrt{1-|y'|^2}} \\ 0 & 1 \end{pmatrix}$

√ (d) $a = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & \frac{-(y')^\intercal}{\sqrt{1-|y'|^2}} \\ \frac{-y'}{\sqrt{1-|y'|^2}} & \frac{1}{1-|y'|^2} \end{pmatrix}, \qquad b = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & 0 \\ \frac{-y'}{\sqrt{1-|y'|^2}} & 1 \end{pmatrix}$

(e) None of the above.

The inverse map $\Psi := \Phi^{-1} \colon B_1 \to \Omega$ is given by $\Psi(y', y^n) = \Big(y', y^n + 1 - \sqrt{1 - |y'|^2}\Big)$. The Jacobi matrices of Φ and Ψ are given by

$$(d\Phi)(x) = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & 0 \\ \frac{-x'}{\sqrt{1-|x'|^2}} & 1 \end{pmatrix}, \qquad (d\Psi)(y) = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & 0 \\ \frac{y'}{\sqrt{1-|y'|^2}} & 1 \end{pmatrix}.$$

Since we have the Euclidean metric $g_{\mathbb{E}}$ on Ω, we equip B_1 with the Riemannian metric

$$h = \Psi^* g_{\mathbb{E}} = (d\Psi)^\intercal \cdot (d\Psi).$$

The equation $-\Delta_{g_{\mathbb{E}}} u = \operatorname{div}_{g_{\mathbb{E}}} f$ in Ω for u implies that $v = u \circ \Psi$ satisfies

$$-\Delta_h v = (\operatorname{div}_{g_{\mathbb{E}}} f) \circ \Psi = \operatorname{div}_h \Big((d\Psi)^{-1} \cdot (f \circ \Psi) \Big) \quad \text{ in } B_1$$

using formula (10.4.3) from the notes. We compute $\sqrt{\det h} = |\det(d\Psi)| \equiv 1$ and

$$\begin{aligned}
\Delta_h v &= \frac{1}{\sqrt{\det h}} \frac{\partial}{\partial y^i} \Big(\sqrt{\det h}\, h^{ij} \frac{\partial v}{\partial y^j} \Big) = \frac{\partial}{\partial y^i} \Big(h^{ij} \frac{\partial v}{\partial y^j} \Big), \\
\operatorname{div}_h X &= \frac{1}{\sqrt{\det h}} \frac{\partial (\sqrt{\det h}\, X^i)}{\partial y^i} = \frac{\partial X^i}{\partial y^i},
\end{aligned}$$

where $X := (d\Psi)^{-1} \cdot (f \circ \Psi)$ and Einstein's summation convention is used. Hence,

$$\begin{aligned}
b &= (d\Psi)^{-1} = (d\Phi \circ \Psi) = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & 0 \\ \frac{-y'}{\sqrt{1-|y'|^2}} & 1 \end{pmatrix}, \\
a &= (h^{ij}) = h^{-1} = (d\Phi \circ \Psi) \cdot (d\Phi \circ \Psi)^{\intercal} \\
&= \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & 0 \\ \frac{-y'}{\sqrt{1-|y'|^2}} & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & \frac{-(y')^{\intercal}}{\sqrt{1-|y'|^2}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & \frac{-(y')^{\intercal}}{\sqrt{1-|y'|^2}} \\ \frac{-y'}{\sqrt{1-|y'|^2}} & \frac{1}{1-|y'|^2} \end{pmatrix}
\end{aligned}$$

where we computed

$$\frac{-y'}{\sqrt{1-|y'|^2}} \cdot \frac{-(y')^{\intercal}}{\sqrt{1-|y'|^2}} + 1 = \frac{|y'|^2}{1-|y'|^2} + 1 = \frac{1}{1-|y'|^2}.$$

Part II. True or false?

12.6. Let $2 \leq n \in \mathbb{N}$, let $\Omega := B_{\frac{1}{2}}(0) \subset \mathbb{R}^n$ and let $g \colon B_r \to \mathbb{R}$ be given by $g(x) = \log\log(\frac{1}{|x|})$. Then, g is in the Morrey space $L^{2,n}(\Omega)$.

(a) True.

√ (b) False.

By definition, any $f \in L^{2,n}(\Omega)$ satisfies $f \in L^2(\Omega)$ and

$$\sup_{x_0\in\Omega,\ 0<r<1} \left(\frac{1}{r^n}\int_{\Omega_r(x_0)} |f|^2\,dx\right) < \infty$$

which implies $f \in L^\infty(\Omega)$ by the Lebesgue differentiation theorem; but $g \notin L^\infty(\Omega)$.

12.7. Let $2 \leq n \in \mathbb{N}$, let $\Omega := B_{\frac{1}{2}}(0) \subset \mathbb{R}^n$ and let $g \colon B_r \to \mathbb{R}$ be given by $g(x) = \log\log(\frac{1}{|x|})$. Then, g is in the Campanato space $\mathcal{L}^{2,n}(\Omega)$.

√ (a) True.

(b) False.

We have $g \in W^{1,n}(\Omega)$ as shown in Beispiel 8.1.2. Moreover, for every $0 < r < 1$,

$$\begin{aligned}
\frac{1}{r^n}\int_{\Omega_r(x_0)} |g - \overline{g}_{x_0,r}|^2\,dx &\leq \frac{1}{r^n}\left(\int_{\Omega_r(x_0)} |g - \overline{g}_{x_0,r}|^n\,dx\right)^{\frac{2}{n}}\left(\int_{\Omega_r(x_0)} 1\,dx\right)^{1-\frac{2}{n}} && (1)\\
&\leq \frac{Cr^{n-2}}{r^n}\left(Cr^n\int_{\Omega_r(x_0)} |\nabla g|^n\,dx\right)^{\frac{2}{n}} && (2)\\
&\leq C\|\nabla g\|^2_{L^n(\Omega)}
\end{aligned}$$

where we applied Hölder's inequality in step (1) and the Poincaré inequality (Satz 8.6.6) in step (2). Note that the constants C do not depend on r nor x_0. Hence,

$$[g]_{\mathcal{L}^{2,n}} = \sup_{x_0\in\Omega,\ 0<r<1}\left(\frac{1}{r^n}\int_{\Omega_r(x_0)} |g - \overline{g}_{x_0,r}|^n\,dx\right)^{\frac{1}{2}} \leq C\|\nabla g\|_{L^n(\Omega)} < \infty.$$

12.8. Let $2 \leq n \in \mathbb{N}$ and $\Omega := B_{\frac{1}{2}} \subset \mathbb{R}^n$. Then, $L^{2,n}(\Omega) \subsetneq \mathcal{L}^{2,n}(\Omega)$.

√ (a) True.

(b) False.

Let $f \in L^{2,n}(\Omega)$. Recall that $\overline{f}_{x_0,r} := \frac{1}{|\Omega_r(x_0)|} \int_{\Omega_r(x_0)} f\, dx$ has the property that

$$\int_{\Omega_r(x_0)} |f - \overline{f}_{x_0,r}|^2\, dx = \min_{a \in \mathbb{R}} \int_{\Omega_r(x_0)} |f - a|^2\, dx \leq \int_{\Omega_r(x_0)} |f|^2\, dx.$$

Therefore $[f]_{\mathcal{L}^{2,n}} \leq \|f\|_{L^{2,n}}$ which implies $L^{2,n}(\Omega) \subset \mathcal{L}^{2,n}(\Omega)$. As shown in the previous questions the inclusion is indeed strict.

12.9. Let $\Omega \subset \mathbb{R}^n$ be any open, bounded domain. Then,

$$\exists C < \infty \quad \forall u \in C^{1,\frac{1}{2}}(\overline{\Omega}) : \quad \|u\|_{C^1(\overline{\Omega})} \leq \tfrac{1}{9}\|u\|_{C^{1,\frac{1}{2}}(\overline{\Omega})} + C\|u\|_{H^1(\Omega)}.$$

√ (a) True.

(b) False.

The embedding $C^{1,\frac{1}{2}}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ is compact (as proven in Satz 8.6.2 using Arzéla–Ascoli) and the embedding $C^1(\Omega) \hookrightarrow H^1(\Omega)$ is continuous. Hence, the claim follows from the interpolation inequality proven in problem 11.5 with $\varepsilon = \frac{1}{9}$.

12.10. Let $\Omega \subset \mathbb{R}^n$ be any bounded domain. Given $0 < \alpha < 1$ let $f \in C^{0,\alpha}(\overline{\Omega})$. Then there exists $g \in C_c^{0,\alpha}(\mathbb{R}^n)$ such that $g|_\Omega = f$.

√ (a) True.

(b) False.

Given $f \in C^{0,\alpha}(\overline{\Omega})$, let $[f]_{C^{0,\alpha}} := \sup\limits_{x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ and consider $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$\tilde{f}(x) = \inf_{y\in\overline{\Omega}} \Big(f(y) + [f]_{C^{0,\alpha}} |y-x|^\alpha \Big).$$

By definition, of $\tilde{f}$ we have $\forall x \in \overline{\Omega}: \ \tilde{f}(x) \leq f(x)$. Moreover, by definition of $[f]_{C^{0,\alpha}}$,

$$\forall x, y \in \overline{\Omega}: \quad f(x) \leq f(y) + [f]_{C^{0,\alpha}} |y-x|^\alpha \qquad \Rightarrow \ \forall x \in \overline{\Omega}: \ f(x) \leq \tilde{f}(x).$$

Therefore, $\tilde{f}|_{\overline{\Omega}} = f$. We claim that $\tilde{f} \in C^{0,\alpha}(\mathbb{R}^n)$. Since $y \mapsto f(y) + [f]_{C^{0,\alpha}}|y-x|^\alpha$ is continuous for any $x \in \mathbb{R}^n$ and $\overline{\Omega}$ compact, the infimum defining $\tilde{f}(x)$ is attained at some $\overline{x} \in \overline{\Omega}$. Let $x, y \in \mathbb{R}^n$ be arbitrary. Assuming $\tilde{f}(x) \geq \tilde{f}(y)$ we have

$$\begin{aligned}
0 \leq \tilde{f}(x) - \tilde{f}(y) &= f(\overline{x}) + [f]_{C^{0,\alpha}} |\overline{x} - x|^\alpha - f(\overline{y}) - [f]_{C^{0,\alpha}} |\overline{y} - y|^\alpha \\
&\leq f(\overline{y}) + [f]_{C^{0,\alpha}} |\overline{y} - x|^\alpha - f(\overline{y}) - [f]_{C^{0,\alpha}} |\overline{y} - y|^\alpha \\
&= [f]_{C^{0,\alpha}} \Big(|\overline{y} - x|^\alpha - |\overline{y} - y|^\alpha \Big) \\
&\leq [f]_{C^{0,\alpha}} \Big(|\overline{y} - x| - |\overline{y} - y| \Big)^\alpha \\
&\leq [f]_{C^{0,\alpha}} |x-y|^\alpha.
\end{aligned}$$

Note that we applied the following lemma: For real values $0 < s \leq t$ and exponents $0 < \alpha < 1$, there holds

$$0 \leq t^\alpha - s^\alpha \leq (t-s)^\alpha.$$

Proof. Since $\frac{s}{t} \in [0,1]$ and $0 < \alpha < 1$, we have $1 - (\frac{s}{t})^\alpha \leq (1 - \frac{s}{t}) \leq (1 - \frac{s}{t})^\alpha$. The claim follows by multiplication with t^α. □

Hence, $\tilde{f} \in C^{0,\alpha}(\mathbb{R}^n)$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfy $0 \leq \varphi \leq 1$, $\varphi|_{\overline{\Omega}} \equiv 1$ and $|\varphi'| \leq 1$. Then, $g := \varphi\tilde{f}$ is compactly supported, satisfies $g|_\Omega = f$ and for every $x, y \in \operatorname{supp} g \subset \operatorname{supp} \varphi$,

$$\begin{aligned}
|g(x) - g(y)| &= |\varphi(x)\tilde{f}(x) - \varphi(x)\tilde{f}(y) + \varphi(x)\tilde{f}(y) - \varphi(y)\tilde{f}(y)| \\
&\leq |\varphi(x)||\tilde{f}(x) - \tilde{f}(y)| + |\varphi(x) - \varphi(y)||\tilde{f}(y)| \\
&\leq [f]_{C^{0,\alpha}} |x-y|^\alpha + |x-y| \|\tilde{f}\|_{C^0(\operatorname{supp}\varphi)} \\
&\leq \Big([f]_{C^{0,\alpha}} + \operatorname{diam}(\operatorname{supp}\varphi)^{1-\alpha} \|\tilde{f}\|_{C^0(\operatorname{supp}\varphi)} \Big) |x-y|^\alpha
\end{aligned}$$

which proves $g \in C_c^{0,\alpha}(\mathbb{R}^n)$.

12.11. Let $\Omega \subset\subset \mathbb{R}^n$ be any open, bounded, smooth domain and let $u \in H^1_0(\Omega)$ be the unique weak solution of $\Delta u = (1 + |x|^2)$ in Ω. Then, $u \geq 0$ in $\overline{\Omega}$.

(a) True.

√ (b) False.

By elliptic regularity, $u \in C^2(\overline{\Omega})$. Moreover, $-\Delta u < 0$ in Ω and $u|_{\partial\Omega} = 0$. Hence, $u \leq 0$ in $\overline{\Omega}$ by the maximum principle. (We also must have $u \not\equiv 0$, thus $u \not\geq 0$.)

12.12. Let $\Omega \subset\subset \mathbb{R}^n$ be any open, bounded, smooth domain and let $u \in H^1_0(\Omega)$ be the unique weak solution of $\Delta u = (1 + |x|^2)$ in Ω. Then, $u(x) \neq 0$ for every $x \in \Omega$.

√ (a) True.

(b) False.

By elliptic regularity, $u \in C^2(\overline{\Omega})$. Moreover, $-\Delta u < 0$ in Ω and $u|_{\partial\Omega} = 0$. Hence, $u \leq 0$ in $\overline{\Omega}$ by the maximum principle. Suppose, $u(x_0) = 0$ at some $x_0 \in \Omega$. Then, u attains an interior maximum at x_0 which by the strong maximum principle implies $u \equiv 0$. This however contradicts $\Delta u = (1 + |x|^2)$.

12.13. Let $B_1 \subset \mathbb{R}^2$ be the unit disc. Let $u \in C^{1,\alpha}(B_1)$ satisfy

$$\forall \varphi \in C_c^\infty(B_1): \quad \int_{B_1} \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1+|\nabla u|^2}}\,dx = 0.$$

Then, $u \in C^2(B_1)$.

√ (a) True.

(b) False.

By assumption, $u \in C^{1,\alpha}(B_1)$ is a weak solution to the (minimal surface) equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{in } B_1.$$

As computed in problem 10.12, this equation reads

$$Lu := \sum_{i,j=1}^{2} a^{ij} u_{ij} = 0, \qquad (a^{ij}) := \begin{pmatrix} 1+u_2^2 & -u_1u_2 \\ -u_1u_2 & 1+u_1^2 \end{pmatrix}$$

in non-divergence form, where the subscripts denote partial derivatives. Since we assume $u \in C^{1,\alpha}(B_1)$, we have $\|\nabla u\|_{C^0(B_1)} < \infty$ and

$$\begin{aligned} \operatorname{trace}(a^{ij}) &= 2 + |\nabla u|^2 \geq 2 > 0, \\ \det(a^{ij}) &= (1+u_2^2)(1+u_1^2) + (u_1u_2)^2 \geq 1 > 0, \end{aligned}$$

which proves that L is uniformly elliptic. Moreover, since $u \in C^{1,\alpha}(B_1)$ by assumption, we have $a^{ij} \in C^{0,\alpha}(B_1)$. Thus, by the interior elliptic regularity theory we have $u \in C^{2,\alpha}$ which implies $u \in C^2(B_1)$.

12.14. If a non-negative function $u \in C^2(\overline{B_1})$ solves $\Delta u = cu$ for some $c \in C^0(\overline{B_1})$ and satisfies $u > 0$ on ∂B_1, then $u > 0$ in B_1.

√ (a) True.

(b) False.

If by contradiction u vanishes somewhere in B_1, then we can find a ball $B \subset\subset B_1$ with $u|_B > 0$ and $u(x_0) = 0$ for some $x_0 \in \partial B$. Indeed, take a ray $y_0 + tv$, with $|v| = 1$ and $t \in [0, \infty[$, starting at a point y_0 where u vanishes and passing through a point where $u > 0$. Then take e. g. t to be the minimum value such that

$$f(t) := \operatorname{dist}(y_0 + tv, \{u = 0\}) - \tfrac{1}{2}\operatorname{dist}(y_0 + tv, \partial B_1)$$

vanishes and notice that $y_0 + tv \in B_1$, since $f(0) < 0$ and $f(T) > 0$, T being the unique time for which $y_0 + Tv \in \partial B_1$. Finally, set $r := \operatorname{dist}(y_0 + tv, \{u = 0\}) > 0$, put $B := B_r(y_0 + tv)$ and call x_0 a point in $\{u = 0\}$ with minimum distance from $y_0 + tv$, so that $x_0 \in \partial B$. Now observe that, even if c can change sign on B, we have

$$-\Delta u + c^+ u = c^- u \geq 0.$$

Hence, $-u$ satisfies the hypotheses of E. Hopf's lemma in B implying $\partial_\nu(-u)(x_0) > 0$ and thus $u(x_0 + \varepsilon\nu) < 0$ for sufficiently small $\varepsilon > 0$, contradicting $u \geq 0$.

12.15. If $u \in H^1(B_1) \cap C^{0,\alpha}_{\mathrm{loc}}(B_1)$ weakly solves $-\Delta u + cu = 0$ in the unit ball $B_1 \subset \mathbb{R}^n$ for some function $c \in C^{6,\alpha}_{\mathrm{loc}}(B_1)$ then $u \in C^{8,\alpha}_{\mathrm{loc}}(B_1)$.

√ (a) True.

(b) False.

Observe that, if $f, g \in C^{0,\alpha}_{\mathrm{loc}}(B_1)$, then $fg \in C^{0,\alpha}_{\mathrm{loc}}(B_1)$ as we can write

$$\frac{|f(x)g(x) - f(y)g(y)|}{|x-y|^\alpha} \leq |f(x)|\frac{|g(x) - g(y)|}{|x-y|^\alpha} + |g(y)|\frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

and all the terms in the right-hand side are bounded when x, y vary in a compact subset of B_1. From this it easily follows that, if $f, g \in C^{k,\alpha}_{\mathrm{loc}}(B_1)$, then $fg \in C^{k,\alpha}_{\mathrm{loc}}(B_1)$ as well. Now we can prove that $u \in C^{k+2,\alpha}_{\mathrm{loc}}(B_1)$ for all $0 \leq k \leq 6$, by induction on k: the base step holds since $cu \in C^{0,\alpha}_{\mathrm{loc}}(B_1)$, so that the interior Schauder estimates give $u \in C^{2,\alpha}_{\mathrm{loc}}(B_1)$. The inductive step is similar: by inductive hypothesis we know that $u \in C^{(k-1)+2,\alpha}_{\mathrm{loc}}(B_1)$, so in particular (being $k \leq 6$) $c, u \in C^{k,\alpha}_{\mathrm{loc}}(B_1)$ and finally $cu \in C^{k,\alpha}_{\mathrm{loc}}(B_1)$, giving $u \in C^{k+2,\alpha}_{\mathrm{loc}}(B_1)$ by interior Schauder estimates.