1 Maximum principle for elliptic operators

1.1 Heuristics

Let $\Omega \subset\subset \mathbb{R}^n$ be an open domain. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\Delta u \geq 0 \quad \text{in } \Omega.$$

Then, we would expect

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Why do we expect this conclusion to be true? There are two cases guiding our intuition:

- In dimension $n = 1$, the condition $\Delta u \geq 0$ is equivalent to u being convex.

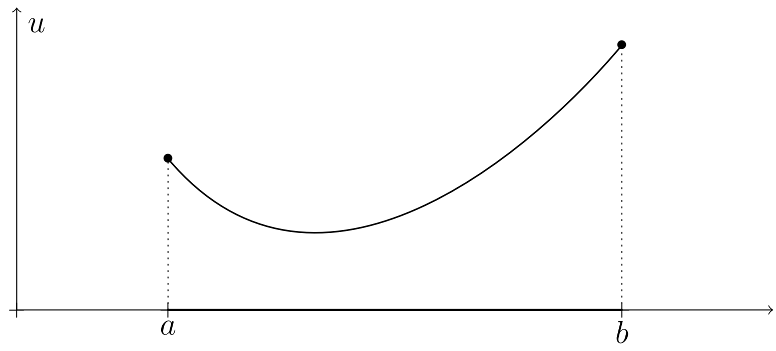

Figure 1: A convex function $u\colon [a, b] \to \mathbb{R}$ attains its maximum on the boundary.

- If $\Delta u = 0$ (i. e. u is harmonic), then u satisfies the mean-value property:

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy$$

for every ball $B_r(x) \subset \Omega$.

If only $\Delta u \geq 0$ in Ω, then it is still true that

$$u(x) \leq ⨍_{B_r(x)} u(y)\, dy$$

when $B_r(x) \subset \Omega$, and thus it is still easy that the claim above holds true. Now, we want to reach the same conclusion for a general class of second-order elliptic operators.

1.2 Setup

Consider the differential operator

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x^j} + c(x)u.$$

We make the following hypotheses:

(1) We assume $a_{ij}, b_j, c \in C^0(\overline{\Omega})$. In fact, boundedness of these coefficients would be sufficient.

(2) We assume uniform ellipticity, so there exists some $\mu > 0$ such that for every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2.$$

Remark. The corresponding results for operators in divergence form, namely

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\Big(a_{ij}(x)\frac{\partial u}{\partial x^j}\Big) + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x^j} + c(x)u$$

shall be recovered as a special case provided we assume (together with uniform ellipticity) that $a^{ij} \in C^1(\overline{\Omega})$. Once again, boundedness of the first derivatives (understood in the classical sense) would be enough.

Theorem 1 (weak maximum principle)**.** *Let $\Omega \subset\subset \mathbb{R}^n$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $Lu \geq 0$ under assumptions (1) and (2).*

- *If $c \leq 0$ in Ω, then $\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u_+$ where $u_+ := \max\{u, 0\}$.*
- *If $c \equiv 0$ in Ω, then $\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u$.*

Remark. The inequality $\max_{\overline{\Omega}} u \geq \max_{\partial\Omega} u$ is obvious, so actually, the second statement of Theorem 1 can be rephrased in the equivalent form $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.

Remark. The maximum principle concerns a second order phenomenon. This will be clear from the proof, but we also present a counterexample for fourth order equations: Let $n = 1$ and $L = \frac{d^4}{dx^4}$. Let $u\colon [0,1] \to \mathbb{R}$ be given by $u(x) = 3x^2 - 4x^3$. So $Lu \equiv 0$ but $x = \frac{1}{2}$ is an interior maximum which would be forbidden by the maximum principle. However, we will see later that there exists an important exception that is represented by holomorphic functions, which are solutions of a first-order problem (the Cauchy-Riemann equations).

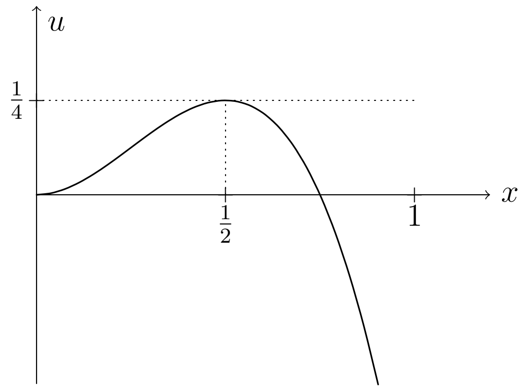

Figure 2: The graph of the function $u(x) = 3x^2 - 4x^3$.

Remark (weak minimum principle). Let $\Omega \subset\subset \mathbb{R}^n$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $Lu \leq 0$ under assumptions (1) and (2).

- If $c \leq 0$ in Ω, then $\min\limits_{\overline{\Omega}} u \geq -\max\limits_{\partial\Omega} u_-$ where $u_- := -\min\{u, 0\}$.
- If $c \equiv 0$ in Ω, then $\min\limits_{\overline{\Omega}} u \geq -\max\limits_{\partial\Omega} u$.

This follows by applying Theorem 1 to $-u$. If $Lu = 0$, then *both* apply and we conclude

$$\max_{\overline{\Omega}} |u| = \max_{\overline{\Omega}} |u|.$$

Moral proof of Theorem 1. If the statement were false, it would mean that u has an *interior* maximum point $x_0 \in \Omega$, where it attains a *positive* value. Together with the first and second derivative test, we obtain

$$\left.\begin{aligned} u(x_0) &> 0 \\ \nabla u(x_0) &= 0 \\ \nabla^2 u(x_0) &\leq 0 \end{aligned}\right\} \qquad \Rightarrow Lu(x_0) \leq 0,$$

where $\nabla^2 u$ denotes the Hessian of u. By assumption, $Lu(x_0) \geq 0$ which is almost a contradiction. To really conclude, we shall introduce a little perturbation to the scope of making one of the inequalities strict. □

Proof of Theorem 1. We argue by contradiction. Suppose,

$$\max_{\overline{\Omega}} u > \max_{\partial\Omega} u_+$$

and set $\delta := (\max_{\overline{\Omega}} u) - (\max_{\partial\Omega} u_+) > 0$. Let $\ell > 0$ be as indicated in figure 3.

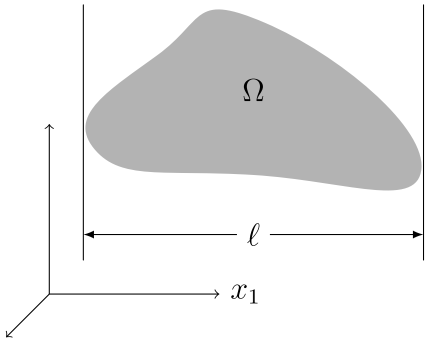

Figure 3: Some domain $\Omega \subset\subset \mathbb{R}^n$.

For any $0 < \varepsilon < \varepsilon_* := \frac{\delta}{2} e^{-\gamma \ell}$ and some parameter $\gamma > 0$ to be chosen, we set $v_\varepsilon = u + \varepsilon e^{\gamma x_1}$. By construction, this still satisfies

$$\max_{\overline{\Omega}} v_\varepsilon > \max_{\partial\Omega} (v_\varepsilon)_+$$

so v_ε has an interior positive maximum point $x_0 \in \Omega$. As before,

$$\left.\begin{aligned} v_\varepsilon(x_0) > 0 \\ \nabla v_\varepsilon(x_0) = 0 \\ \nabla^2 v_\varepsilon(x_0) \leq 0 \end{aligned}\right\} \qquad \Rightarrow L v_\varepsilon(x_0) \leq 0.$$

But

$$L v_\varepsilon(x_0) = \underbrace{Lu(x_0)}_{\geq 0} + \varepsilon(a_{11}\gamma^2 + b_1\gamma + c)e^{\gamma x_1}.$$

By ellipticity, $a_{11} > 0$. For $\gamma \geq \gamma_*$ independently of ε we obtain $a_{11}\gamma^2 + b_1\gamma + c > 0$. With this choice, $L v_\varepsilon(x_0) > 0$.

In the case $c = 0$, the inequality $Lu \geq 0$ implies $L(u + \kappa) \geq 0$ for any constant $\kappa \in \mathbb{R}$. So pick $\kappa \in \mathbb{R}$ large so that $u + \kappa > 0$ in $\overline{\Omega}$. In particular, $(u + \kappa) = (u + \kappa)_+$. Thus,

$$\max_{\overline{\Omega}} (u + \kappa) = \max_{\partial\Omega} (u + \kappa)$$

and we conclude by subtracting κ from both sides of the equality. □

Comment. It is still possible that there exists $x_0 \in \Omega$ such that $\max_{\overline{\Omega}} u = u(x_0)$. The strong maximum principle will state that this *cannot* happen unless u is constant.

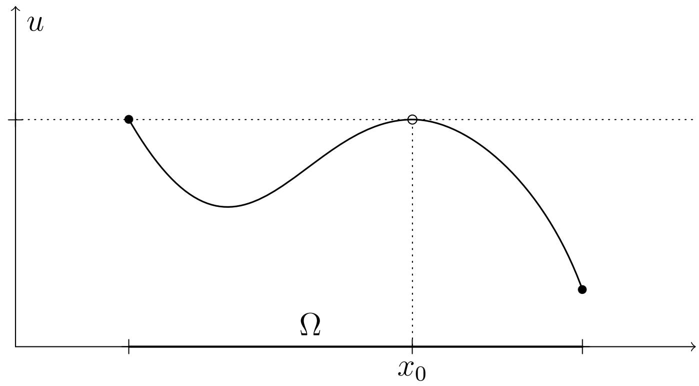

Figure 4: Some interior maximum not exceeding the maximum on the boundary.

1.3 Inhomogeneous Case

Theorem 2. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ solve $Lu = f$ under conditions (1) and (2). If $c \leq 0$, then*

$$\max_{\overline{\Omega}} |u| \leq \max_{\overline{\Omega}} |u| + \kappa \sup_{\Omega} |f| \tag{$*$}$$

where $\kappa = \mu^{-1} e^{2(1+\beta)} d^2$ given the ellipticity constant μ, the diameter $d := \operatorname{diam}(\Omega)$ and $\beta := \sup_\Omega \mu^{-1}(d|b_1| + d^2|c|)$.

Proof. Without loss of generality (by scaling), we can assume $\operatorname{diam}(\Omega) = 1$ and $\mu = 1$. So we need to prove $(*)$ with $\kappa = e^{2(1+\beta)}$. Set

$$v := u + \Big(\sup_{\Omega} |f|\Big)\Big(e^{\gamma(x_1+1)} - 1\Big).$$

The goal is to choose $\gamma \in \mathbb{R}$ so that $Lv \geq 0$. By linearity of L,

$$Lv = \underbrace{Lu}_{f} + \Big(\sup_{\Omega} |f|\Big)\Big(a_{11}\gamma^2 + b_1\gamma + c(1 - e^{-\gamma(x_1+1)})\Big)e^{\gamma(x_1+1)}.$$

The second term will dominate the first, provided $a_{11}\gamma^2 + b_1\gamma + c(1 - e^{-\gamma(x_1+1)}) \geq 1$ and $1 < e^\gamma \leq e^{\gamma(x_1+1)} \leq e^{2\gamma}$. So it is enough to pick $\gamma > 0$ such that

$$a_{11}\gamma^2 - |b_1|\gamma - |c| \geq 1.$$

This is possible since $a_{11} \geq 1$ (as we assume $\mu = 1$) and, in fact, $\gamma = 1 + \beta = (1 + |b_1| + |c|)$ works. So v satisfies the (homogeneous) weak maximum principle. As a result,

$$\max_{\overline{\Omega}} |u| \leq \max_{\overline{\Omega}} v \leq \max_{\partial\Omega}(v_+) \leq \Big(\max_{\partial\Omega} |u|\Big) + \Big(\sup_{\Omega} |f|\Big) e^{2(1+\beta)}$$

where we have $|u|$ on the left-hand side as the argument applies to u and $-u$ alike. $\square$

Corollary. *Let the hypotheses be as in Theorem 2 except that we replace the assumption* $c \leq 0$ *by the condition* $\mu^{-1}e^{2(1+\beta)}d^2(\sup c_+) \leq \varepsilon < 1$*. Then,*

$$\max_{\overline{\Omega}}|u| \leq (1-\varepsilon)^{-1}\max_{\partial\Omega}|u| + \mu^{-1}(1-\varepsilon)^{-1}e^{2(1+\beta)}d^2\sup_{\Omega}|f|.$$

Proof. Since $cu = (c_+ - c_-)u$, the equation $Lu = f$ reads

$$L_1 u := \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{j=1}^{n} b_j \frac{\partial u}{\partial x^j} - c_- u = f - c_+ u =: \tilde{f}.$$

Applying Theorem 2 to the equation $L_1 u = \tilde{f}$, we obtain

$$\begin{aligned}\max_{\overline{\Omega}}|u| &\leq \max_{\overline{\Omega}}|u| + \kappa\sup_{\Omega}|\tilde{f}| \\ &\leq \max_{\overline{\Omega}}|u| + \kappa\Big(\sup_{\Omega}|f| + \Big(\sup_{\Omega}c_+\Big)\Big(\max_{\overline{\Omega}}|u|\Big)\Big).\end{aligned}$$

By hypothesis, $\kappa(\sup_\Omega c_+) \leq \varepsilon$, hence

$$(1-\varepsilon)\max_{\overline{\Omega}}|u| \leq \max_{\partial\Omega}|u| + \kappa\sup_{\Omega}|f|.$$ □

1.4 Connection to Complex Analysis

Like we anticipated above, there is an important exception to the principle that the maximum principle captures a second-order phenomenon.

Consider the unit disk $\mathbb{D} \subset \mathbb{R}^2 \simeq \mathbb{C}$ and a function $f \in C^1(\mathbb{D};\mathbb{C})$. We write $\partial_{\overline{z}} f = 0$ if f satisfies the Cauchy–Riemann equations

$$\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)f = 0$$

which is equivalent to $f\colon \mathbb{D} \to \mathbb{C}$ being holomorphic. Then, f satisfies a maximum modulus principle, i. e. $\sup_{B_1}|f|$ cannot be attained in the interior. This is already a strong maximum principle.

Lemma 1 (Schwarz lemma)**.** *Let* $\mathbb{D} \subset \mathbb{C}$ *be the unit disk. Let* $f \in \mathrm{Aut}(\mathbb{D})$*, i. e. let* $f\colon \mathbb{D} \to \mathbb{C}$ *be holomorphic satisfying* $f(\mathbb{D}) \subset \mathbb{D}$*. If* $f(0) = 0$*, then*

$$\forall z \in \mathbb{D}: \quad |f(z)| \leq |z|$$

with rigidity in the equality case.

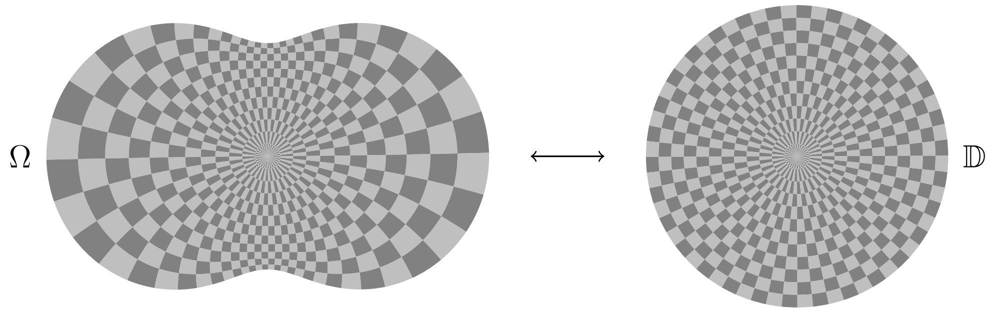

Figure 5: bi-holomorphic map between a domain $\Omega \subset \mathbb{C}$ and the unit disc $\mathbb{D} \subset \mathbb{C}$.

The Schwarz lemma implies the Riemann mapping theorem: Let $\Omega \subset \mathbb{C}$ be simply connected open domain. Then it is bi-holomorphic to the unit disc.

The Riemann mapping theorem in turn implies the Uniformisation theorem: Let S be any Riemann surface. Then it is bi-holomorphic to one of the models given in figure 6.

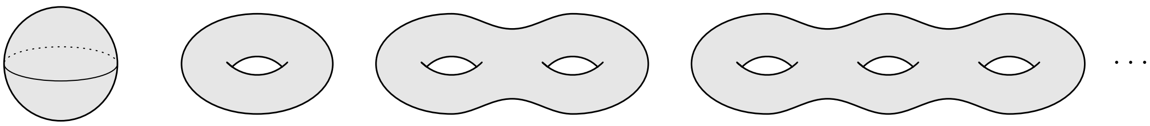

Figure 6: model surfaces

If $f\colon \Omega \to \mathbb{C}$ is holomorphic, then $\mathrm{Re}(f)$ is harmonic. Properties of holomorphic functions can be transformed to properties of harmonic functions and viceversa, which allows to build a bridge between the two worlds.