Sample Questions 1

- For what exponents p does $\{0\} \subset \mathbb{R}^n$ have vanishing $W^{1,p}$-capacity?
- What is an elliptic operator? Give examples and non-examples.
- Prove existence of solutions $u \in C^{2,\alpha}$ of the elliptic equation

  $$\sum \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + cu = f$$

  provided that solutions exist in the case $c = 0$.
- Discuss existence of weak derivatives of the functions

  $$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto |x| \end{aligned} \qquad\qquad \begin{aligned} g\colon \mathbb{R} &\to \mathbb{R}. \\ x &\mapsto \tfrac{x}{|x|} \end{aligned}$$

Sample Questions 2

- Define the Sobolev space $W^{1,p}(\Omega)$ and discuss its properties.
- State the spectral theorem for the Laplace operator on a bounded domain Ω. Explain why the operator $K\colon L^2(\Omega) \to L^2(\Omega)$ mapping $f \in L^2(\Omega)$ to the weak solution u of $-\Delta u = f$ in Ω is compact.
- What can you say about the regularity of the eigenfunctions of $-\Delta$ in the interior and on the boundary of the domain?
- Let $B_R \subset \mathbb{R}^n$ be the ball of radius $R > 0$. For $p < n$ and $q \le \frac{np}{n-p}$ consider

  $$\|f\|_{L^q(B_R)} \le CR^\alpha \|\nabla f\|_{L^p(B_R)}$$

  and compute α by scaling.

Sample Questions 3

- Define $H^1_0(\Omega)$. Characterise $H^1_0(\Omega)$ in terms of trace operators.
- Prove the Poincaré inequality for $H^1_0(\Omega)$. Do you know any similar inequality without null boundary conditions?
- Prove E. Hopf's boundary point lemma.
- Consider the Hamiltonian for the quantum oscillator on the interval $]-1,1[$ i. e. $L = -\Delta + V$ where $0 \le V \in C^0(]-1,1[)$ (for instance $V(x) := \frac{1}{2}|x|^2$). What can you say about its spectrum?