Exercise Sheet 1

Please hand in your solutions by January 26, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

All the definitions needed for intrinsic manifolds can be found in Chapter 2.8 of the lecture notes from last semester.

1. In this exercise, we will introduce projective spaces. They are an important class of differential manifolds.

a) The real projective plane $\mathbb{R}P^2$ is the set

$$\mathbb{R}P^2 = \{\ell \subset \mathbb{R}^3 \,:\, \ell \text{ is a 1-dimensional linear subspace}\}$$

of real lines in $\mathbb{R}^3$. It can be identified with the quotient space

$$\mathbb{R}P^2 = \left(\mathbb{R}^3 \setminus \{0\}\right)/\mathbb{R}^*$$

of nonzero vectors in $\mathbb{R}^3$ modulo the action of the multiplicative group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ of nonzero real numbers. The equivalence class of a nonzero vector $x = (x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{0\}$ will be denoted by

$$[x] = [x_0 : x_1 : x_2] := \{\lambda x : \lambda \in \mathbb{R}^*\}.$$

We define subsets $U_i = \{[x] \in \mathbb{R}P^2 : x_i \neq 0\}$ and charts $\phi_i : U_i \to \mathbb{R}^2$ by $[x] \mapsto (x_j/x_i, x_k/x_i)$ for $i = 0, 1, 2$ and where $\{i, j, k\} = \{0, 1, 2\}$, $j < k$.

(i) Check that $\mathbb{R}P^2$ with the atlas $\{\phi_i\}_{i=0,1,2}$ is a smooth manifold of dimension 2.

(ii) Check that the intrinsic manifold topology is the quotient topology of $\mathbb{R}P^2$

Remark: Note how this definition is more natural than the one seen in exercise 4 of sheet 2 of the first semester.

b) The complex projective space $\mathbb{C}P^1$ is the set

$$\mathbb{C}P^1 = \{\ell \subset \mathbb{C}^2 \,:\, \ell \text{ is a 1-dimensional complex linear subspace.}\}$$

of complex lines in $\mathbb{C}^2$. It can be identified with the quotient space

$$\mathbb{C}P^1 = \left(\mathbb{C}^2 \setminus \{0\}\right)/\mathbb{C}^*$$

of nonzero vectors in $\mathbb{C}^2$ modulo the action of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of nonzero complex numbers. The equivalence class of a nonzero vector $z = (z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}$ will be denoted by

$$[z] = [z_0 : z_1] := \{\lambda z : \lambda \in \mathbb{C}^*\}.$$

(i) Find an atlas, which makes $\mathbb{C}P^1$ into a smooth manifold.

(ii) Find an explicit diffeomorphism between $\mathbb{C}P^1$ and the sphere S^2.

Remark: $\mathbb{R}P^n$ and $\mathbb{C}P^n$ can be defined similarly for all $n \geq 0$.

2. **a)** We define $\varphi_1 : \mathbb{R} \to \mathbb{R} : x \mapsto x$ and $\varphi_2 : \mathbb{R} \to \mathbb{R} : x \mapsto x^3$. Prove that $\mathcal{A}_i = \{\varphi_i\}$ are smooth atlases for $\mathbb{R}$ with $i = 1, 2$. Prove that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is not an atlas. Prove that $(\mathbb{R}, \mathcal{A}_1)$ is diffeomorphic to $(\mathbb{R}, \mathcal{A}_2)$.

Remark: It can be proven that in dimension 1,2 and 3, every underlying topological space of a smooth manifold has only one smooth manifold structure up to diffeomorphism. It is an open question whether S^4 has a unique differentiable structure. A remarkable result by Donaldson give uncountably many non-diffeomorphic smooth structures on $\mathbb{R}^4$. Another result by Milnor says that S^7 allows exactly 28 different smooth structures up to diffeomorphisms.

b) Prove that any submanifold M of $\mathbb{R}^n$ as seen in the last semester is also an intrinsic manifold.

Remark: The Whitney Embedding Theorem gives a converse to this statement. Namely, for every intrinsic m-manifold M, there is an embedding as submanifold into $\mathbb{R}^n$ for $n = 2m$. So intrinsic manifolds are the same concept as submanifolds in $\mathbb{R}^n$ from last semester. However this semester, in the context of differential topology, we only want to study properties of manifolds up to diffeomorphisms and not manifolds and their specific embedding into $\mathbb{R}^n$. This is why we need the more abstract concept of intrinsic manifolds.

3. **a)** We identify $\mathbb{C}P^1$ with $\mathbb{C} \cup \{\infty\}$ by $[z : 1] \mapsto z$ and $[1 : 0] \mapsto \infty$. Consider the map $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ given by $z \mapsto \frac{1}{z}$ for $z \neq 0$ and $0 \mapsto \infty$. Prove that this defines a smooth diffeomorphism $f : \mathbb{C}P^1 \to \mathbb{C}P^1$.

b) Prove that the map $g : S^2 \to \mathbb{R}P^2 : (x_0, x_1, x_2) \mapsto [x_0 : x_1 : x_2]$ is a smooth double cover.

4. Let M be a manifold with an atlas $\mathcal{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$. We give the following two definitions of the tangent space at $p \in M$.

(i) Two smooth curves $\gamma_0, \gamma_1 : \mathbb{R} \to M$ with $\gamma_0(0) = \gamma_1(0) = p$ are called p-equivalent if for some (and hence every) $\alpha \in A$ with $p \in U_\alpha$ we have

$$\frac{d}{dt}\bigg|_{t=0} \phi_\alpha(\gamma_0(t)) = \frac{d}{dt}\bigg|_{t=0} \phi_\alpha(\gamma_1(t)).$$

We write $\gamma_0 \stackrel{p}{\sim} \gamma_1$ if γ is p-equivalent to γ_1 and denote the equivalence class of a smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$ by $[\gamma]_p$. The tangent space of M at p is the set of equivalence classes

$$T_pM := \{[\gamma]_p : \gamma : \mathbb{R} \to M \text{ is smooth and } \gamma(0) = p\}.$$

(ii) The $\mathcal{A}$-tangent space of M at p is the quotient space

$$T_p^{\mathcal{A}}M := \bigcup_{p \in U_\alpha} \{\alpha\} \times \mathbb{R}^m / \sim_p,$$

where the union runs over all $\alpha \in A$ with $p \in U_\alpha$ and

$$(\alpha, \xi) \sim_p (\beta, \eta) \iff d(\phi_\beta \circ \phi_\alpha^{-1})(x)\xi = \eta, x := \phi_\alpha(p).$$

The equivalence class will be denoted by $[\alpha, \xi]_p$.

Show that the natural map

$$T_pM \mapsto T_p^{\mathcal{A}}M : [\gamma]_p \mapsto \left[\alpha, \frac{d}{dt}\bigg|_{t=0} \phi_\alpha(\gamma(t))\right]$$

is well-defined and bijective.

Since $T_p^{\mathcal{A}}M$ has a canonical vector space structure of dimension m, this bijection induces a vector space structure on the set T_pM. This exercise shows that both definitions of tangent space are equivalent.

5. Let M be a m-manifold with an atlas $\mathcal{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$.

a) Define the tangent bundle as

$$TM = \bigcup_{p \in M} \{p\} \times T_pM,$$

and denote by $\pi : TM \to M$ the projection given by $\pi(p, v) := p$.

Prove that TM is a smooth $2m$-dimensional manifold with atlas consisting of the charts

$$\tilde{\phi}_\alpha : \tilde{U}_\alpha := \pi^{-1}(U_\alpha) \to \phi_\alpha(U_\alpha) \times \mathbb{R}^m : (p, v) \mapsto (\phi_\alpha(p), d\phi_\alpha(p)v)$$

for $\alpha \in A$. Also, verify that $\pi : TM \to M$ is a smooth submersion.

b) We give the following two definitions of vector fields.

(i) A map $X : M \to TM$ is a vector field if X is smooth and $\pi \circ X = \mathrm{id}$. Denote the set of such maps X by $\mathrm{Vect}(M)$.

(ii) A collection of smooth maps $X_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^m$ for $\alpha \in A$ is called $\mathcal{A}$-vector field if for all $\alpha, \beta \in A$ and all $x \in \phi_\alpha(U_\alpha \cap U_\beta)$, we have $X_\beta(\phi_{\beta\alpha}(x)) = d\phi_{\beta\alpha}(x)X_\alpha(x)$. Denote the set of such collections $\{X_\alpha\}_{\alpha\in A}$ by $\mathrm{Vect}^{\mathcal{A}}(M)$.

Define the map $\mathrm{Vect}(M) \to \mathrm{Vect}^{\mathcal{A}}(M)$ by

$$X \mapsto \{X_\alpha\}_{\alpha\in A} \text{ where } X_\alpha(x) := d\phi_\alpha(p)X(p),\ p := \phi_\alpha^{-1}(x).$$

Prove that this map is well-defined and bijective.

6. **a)** Prove that there exists a canonical isomorphism $T_\ell \mathbb{R}P^n \cong \mathcal{L}(\ell, \ell^\perp)$ where $\perp$ is the orthogonal complement with respect to the Euclidean metric on $\mathbb{R}^{n+1}$.

b) Prove that the tangent bundle $T\mathbb{T}^3$ of the three torus is diffeomorphic to $\mathbb{T}^3 \times \mathbb{R}^3$. Characterise vector fields on $\mathbb{T}^3$.