Exercise Sheet 2

Please hand in your solutions by March 12, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. If $m < p$, show that every smooth map $f : M^m \to S^p$ is homotopic to a constant map.

2. If two maps f and g from M to S^k satisfy $\|f(x) - g(x)\| < 2$ for all x, prove that f is homotopic to g, the homotopy being smooth if f and g are smooth.

3. Prove that any compact manifold M^k can be embedded into R^{2k+1}.

Hint: Use or reprove the fact that M can be embedded into $\mathbb{R}^N$ for some large k, $f : M \hookrightarrow \mathbb{R}^N$. Prove that the projection parallel to v, $\|v\| = 1$, $\pi_v : \mathbb{R}^N \to v^\perp = H$, $\pi_v(x) = x - \langle x, v\rangle v$ induces an embedding of $f(M)$, for some v. To prove this use Sard's theorem with the following auxiliary maps:

$$F : M \times M \times \mathbb{R} \to \mathbb{R}^N,\quad F(p, q, t) = t(f(p) - f(q)),\quad G : TM \times \mathbb{R} \to \mathbb{R}^N,\quad G(p, v, t) = tdf(p)v.$$

4. Let $M = S^{k_1} \times \cdots \times S^{k_\ell}$ be a product of spheres. This a smooth manifold of dimension $m = k_1 + \cdots + k_\ell$. Show that M can be embedded into $\mathbb{R}^{m+1}$.

Hint: Prove the following (stronger) statement by induction over ℓ: There exists an embedding $S^{k_1} \times S^{k_2} \cdots \times S^{k_\ell} \times \mathbb{R} \hookrightarrow \mathbb{R}^{m+1}$ where $m = k_1 + \cdots + k_\ell$ as before.

5. Define $f : \mathbb{R}P^2 \to \mathbb{R}^4$ by

$$f([x : y : z]) = \frac{1}{x^2 + y^2 + z^2}(x^2 - y^2, xy, xz, yz)$$

a) Show that f is injective

b) Show that f is an immersion

c) Show that f induces an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$.

6. **a)** Let M_1 be a manifold with boundary and let M_2 be a manifold without boundary. Prove that $M_1 \times M_2$ is a manifold with boundary.

b) If M_1 and M_2 are manifolds with boundary, is $M_1 \times M_2$ a manifold with boundary?

c) Prove that if $\partial M = \emptyset$, then M is a boundary, i.e. there exist W with $\partial W = M$.

d) Let M be a manifold without boundary. Prove that if $a < b$ are two regular values of $f : M \to \mathbb{R}$, then $N = f^{-1}([a, b])$ is a manifold with boundary.

e) Let $P := \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1, 0 \le x \le 2\}$ which is a portion of a torus and called pair of pants. Prove that this is a manifold with boundary.