Exercise Sheet 3

Please hand in your solutions by March 12, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. Let $f : S^1 \to S^1$ be a smooth map.

a) Show that there exists $k \in \mathbb{Z}$ and a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ with

$$\phi(x+1) = \phi(x) + k \qquad \text{and} \qquad f(\exp(2\pi \mathbf{i} x)) = \exp(2\pi \mathbf{i} \phi(x))$$

for all $x \in \mathbb{R}$.

b) Show that $\deg(f) = k$, where $k \in \mathbb{Z}$ is as in part (a) .

Hint: For b): Show first that f is homotopic to $z \mapsto z^k$. For this part a) might come in handy.

2. [**Existence of a Morse function**] Let $M^m \subset \mathbb{R}^n$ be a smooth manifold and $f : M \to \mathbb{R}$ be a smooth function. The Hessian of f at $p \in M$ is the bilinear map

$$H_p f : T_p M \times T_p M \to \mathbb{R}$$

defined by the covariant derivative of df, i.e.

$$H_p f(X, Y) := \nabla_X (df)_p(Y) := \mathcal{L}_X(df(p)Y) - df(p)\nabla_X Y.$$

This is always symmetric and does not depend on the embedding when $df(p) = 0$ is a critical point. We call f a **Morse function**, when $H_p f$ is nondegenerate for all $p \in M$ with $df(p) = 0$.

Denote by $TM^\perp := \{(p, v) \,|\, p \in M,\, v \in T_p M^\perp\}$ the normal bundle of M and define

$$\phi : TM^\perp \to \mathbb{R}^n, \qquad \phi(p, v) = p + v.$$

a) Prove that $x \in \mathbb{R}^n$ is a regular value of ϕ if and only if the function

$$f_x : M \to \mathbb{R}, \qquad f_x(p) := \frac{1}{2}\|p - x\|^2$$

is a Morse function on M.

b) Prove that there exists a Morse function on every manifold M.

Remark: Morse functions are an important tool in differential topology: They are used to construct various invariants of manifolds, to classify closed 2-manifolds and to prove the famous h-cobordism theorem in higher dimensions.

3. Let $f : M \to N$ and $g : N \to Q$ be smooth maps between m-dimensional compact connected manifolds. Show that $\deg(g \circ f) = \deg(f) \cdot \deg(g)$.

4. Show that every smooth map $f : S^n \to S^n$ of degree different from $(-1)^{n+1}$ must have a fixed point.

Hint: Suppose f has no fixed point. Then there exists a homotopy between f and the antipodal map $S^n \to S^n, x \mapsto -x$.

5. **a)** Suppose $g : S^n \to S^n$ satisfies $g(x) = g(-x)$ for all $x \in S^n$. Then the degree of g is even.

b) Suppose $f : S^n \to S^n$ is a smooth function with odd degree. Then f carries some pair of antipodal points into a pair of antipodal points.

6. **a)** Show that the tangent bundle TM of a smooth manifold M is always an orientable manifold.

b) * Show that every simply connected, connected manifold M is orientable.

Hint: For a): Have a closer look at the atlas for TM discussed in Exercise Sheet 1, Exercise 5. For b): Fix $p \in M$ and a frame $e : \mathbb{R}^m \to T_pM$. For any point $q \in M$ choose a smooth path $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$ and choose a smooth lift $\beta(t) = (\gamma(t), e(t))$ in the frame bundle. We decree that $e(1) : \mathbb{R}^m \to T_qM$ is orientation preserving. Now verify (1) this does not depend on the choices made for γ and β and (2) these orientations on the tangent spaces T_qM "fit smoothly together" and define an orientation on M.