Exercise Sheet 4

Please hand in your solutions by March 19, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. Let $M^m \subset \mathbb{R}^n$ be a smooth, compact manifold without boundary and for $\epsilon > 0$ denote

$$N_\epsilon := \left\{ p + v \,|\, p \in M,\, v \in T_pM^\perp,\, \|v\| \leq \epsilon \right\}.$$

a) Prove that for $\epsilon > 0$ sufficiently small N_ϵ is a smooth manifold with boundary.

b) Prove that there exists a unique smooth map $r : N_\epsilon \to M$ that satisfies

$$\|x - r(x)\| = \min_{p \in M} \|x - p\|.$$

Show for all $x \in N_\epsilon$ that $x - r(x) \perp T_{r(x)}M$.

Hint: For a): Show that $\phi : TM^\perp \to \mathbb{R}^n$, $\phi(p, v) := p + v$ restricts to an embedding on a suitable neighborhood of $M \times \{0\} \subset TM^\perp$. For b): It holds $r(p + v) = p$. Uniqueness of the map r follows from injectivity of ϕ.

2. Let M be a m-dimensional oriented compact manifold with boundary and let $f : M \to \mathbb{R}^m$ be a smooth map. For a regular value $y \in \mathbb{R}^m \setminus f(\partial M)$ define

$$g_y : \partial M \to S^{m-1}, \qquad g_y(p) := \frac{f(p) - y}{\|f(p) - y\|}.$$

Prove that

$$\deg(g_y) = \deg(f, y) = \sum_{p \in f^{-1}(y)} \operatorname{sign}(df(p)).$$

Hint: The modulo 2 version of this result has been proven in Chapter VI, Lemma 7 of the lecture. The same argument works in general, but you have to keep track of the orientations.

3. Given disjoint manifolds $M, N \subset \mathbb{R}^{k+1}$, the linking map is defined by

$$\lambda : M \times N \to S^k, \qquad \lambda(x, y) := \frac{x - y}{\|x - y\|}.$$

If M^m and N^n are compact, oriented, and boundaryless, with total dimension $m + n = k$, then the degree of λ is called the *linking number* $\ell(M, N)$.

a) Prove that $\ell(N, M) = (-1)^{(m+1)(n+1)} \ell(M, N)$.

b) If M bounds an oriented manifold X disjoint from N, then $\ell(M, N) = 0$.

c) Define the linking number for disjoint submanifolds $M^m, N^n \subset S^{k+1}$ for $m, n \geq 1$.

Hint: For c): You may use the fact that $S^{m+n+1} \backslash (N \cup M)$ is nonempty and path-connected for $m, n \geq 1$ without proof.

4. Let M, N be compact manifolds. Prove that the Euler characteristic χ satisfies the following:

a) $\chi(M \times N) = \chi(M) \cdot \chi(N)$.

b) $\chi(M \cup N) = \chi(M) + \chi(N)$ when M and N are disjoint.

c) $\chi(S^{2k+1}) = 0$ and $\chi(S^{2k+2}) = 2$ for all $k \in \mathbb{N}$.

d) $\chi(T^2) = 0$.

5. Let $P, Q \in \mathbb{C}[x]$ be polynomials of degrees $d_P, d_Q > 0$ over $\mathbb{C}$ where P and Q have no common zeroes.

a) Define the degree for the map $P : \mathbb{C} \to \mathbb{C}$ in the usual way and check that even though $\mathbb{C}$ is not compact, the degree is independent from the regular value.

b) Define the map $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ by $f([z : 1]) = [P(z), 1]$ for $z \in \mathbb{C}$ and $f([1 : 0]) = [1 : 0]$. Prove that f is smooth and that it has degree d_P.

c) Define the map $g : \mathbb{C}P^1 \to \mathbb{C}P^1$ by $g([z : w]) = [w^d P(z/w) : w^d Q(z/w)]$ where $d = \max(d_P, d_Q)$. Prove that g is a smooth map and that it has degree d.

d) Give a map $h_k : \mathbb{C}P^1 \to \mathbb{C}P^1$ of degree k for all $k \in \mathbb{Z}$.

6. **a)** * Let $M^m \subset \mathbb{R}^{m+1}$ be a smooth, compact, connected manifold without boundary. Prove that M divides $\mathbb{R}^{m+1}$ into two components. The bounded component is called the interior of M and the unbounded component the exterior of M.

Hint: Define for $y \in \mathbb{R}^{m+1} \setminus M$ the map $g_y : M \to S^m : x \mapsto \frac{x-y}{|x-y|}$ and $d_y := \deg_2(g_y)$. Define two sets $V_0 := \{y \in \mathbb{R}^{m+1} \setminus M : d_y = 0\}$ and $V_1 := \{y \in \mathbb{R}^{m+1} \setminus M : d_y = 1\}$. Prove that these are the two path connected components and find out which one is unbounded.

b) Prove that a smooth, compact hypersurface $M \subset \mathbb{R}^n$ is orientable.

c) Prove that $\mathbb{R}P^2$ cannot be embedded into $\mathbb{R}^3$.