Exercise Sheet 5

Please hand in your solutions by March 26, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. **a)** Find a surjective map $f : S^1 \to S^1$ with $\deg(f) = 0$.

b) For every $k \in \mathbb{Z}$ and $m \geq 1$ find a map $g_k : S^m \to S^m$ with $\deg(g_k) = k$.

2. Let M be a compact, connected, non-orientable manifold without boundary.

a) Let $p, q \in M$ be two points and choose bases $X = (X_1, \dots, X_m) \subset T_pM$ and $X' = (X'_1, \dots, X'_m) \subset T_qM$. Show that (p, X) and (q, X') are framed cobordant.

b) Show that two framed 0-dimensional submanifolds $(N = \{p_1, \dots, p_\ell\}, X)$ and $(N' = \{q_1, \dots, q_k\}, X')$ are framed cobordant if and only if the number of points $\#N$ and $\#N'$ agrees modulo two.

c) Show that two maps $f, g : M \to S^m$ are homotopic if and only if they have the same mod 2 degree.

Hint: Part a) uses similar ideas as the solution of Exercise 6 b) of Exercise Sheet 3.

3. Let M be a compact, connected, oriented manifold without boundary.

a) Let $p, q \in M$ be two points and choose a bases $X = (X_1, \dots, X_m) \subset T_pM$ and $X' = (X'_1, \dots, X'_m) \subset T_qM$. Show that (p, X) and (q, X') are framed cobordant if and only if X and X' are both positive or both negative bases.

b) Show that two framed 0-dimensional submanifolds $N = (\{p_1, \dots, p_\ell\}, X)$ and $N' = (\{q_1, \dots, q_k\}, X')$ are framed cobordant if and only if the number of points counted with signs

$$\sum_{i=1}^{\ell} \operatorname{sign}(X, p_i) = \sum_{j=1}^{k} \operatorname{sign}(X', q_j)$$

agrees for both framed submanifolds.

c) Deduce Hopf's theorem: Two maps $f, g : M \to S^m$ are homotopic if and only if they have same degree.

Hint: Part a) uses similar ideas as the solution of Exercise 6 b) of Exercise Sheet 3.

4. Let M and N be the following two circles in $\mathbb{R}^3$

$$M = \{x^2 + y^2 = 1,\ z = 0\}, \qquad N = \{(x-1)^2 + z^2 = 1,\ y = 0\}.$$

Fix orientations of M and N and compute the linking number $\ell(M,N)$ for these orientations. (See Exercise 4, Serie 4)

5. The Hopf invariant. If $y \neq z$ are regular values for a map $f : S^{2n-1} \to S^n$ for $n \geq 2$, then the manifolds $f^{-1}(y)$ and $f^{-1}(z)$ can be oriented. Hence the linking linking number $\ell(f^{-1}(y), f^{-1}(z))$ is defined. (See exercise 4 c) of sheet 4).

a) Explain how an orientation for $f^{-1}(y)$ and $f^{-1}(z)$ is obtained.

b) Prove that the linking number is locally constant as a function of y.

c) If y and z are regular values of $g : S^{2n-1} \to S^n$ also, where

$$\|f(x) - g(x)\| < \|y - z\|$$

for all x, prove that

$$\ell(f^{-1}(y), f^{-1}(z)) = \ell(g^{-1}(y), f^{-1}(z)) = \ell(g^{-1}(y), g^{-1}(z)).$$

d) Prove that the linking number $\ell(f^{-1}(y), f^{-1}(z))$ depends only on the homotopy class of f, and does not depend on the choice of y and z.

This integer $H(f) := \ell(f^{-1}(y), f^{-1}(z))$ is called the **Hopf invariant** of f.

6. **a)** If the dimension n is odd, prove that $H(f) = 0$.

b) For a composition

$$S^{2n-1} \xrightarrow{f} S^n \xrightarrow{g} S^n$$

prove that $H(g \circ f) = deg(g)^2 \cdot H(f)$.

c) The **Hopf fibration** $\pi : S^3 \to S^2$ is defined as the composition of

$$S^3 = \Big\{(z_1, z_2) \in \mathbb{C}^2 \ :\ |z_1|^2 + |z_2|^2 = 1\Big\} \to \mathbb{C}P^1, \quad (z_1, z_2) \mapsto [z_1 : z_2].$$

with the diffeomorphism $\Phi : \mathbb{C}P^1 \to S^2$ induced by stereographic projection (see Exercise Sheet 1, Exercise 1). Prove that

$$\pi(z_1, z_2) = \Big(2\mathrm{Re}(z_1\overline{z_2}),\ \ 2\mathrm{Im}(z_1\overline{z_2}),\ \ |z_2|^2 - |z_1|^2\Big) \in S^2 \subset \mathbb{R}^3.$$

d) Prove that $H(\pi) = 1$, where π is the Hopf fibration in c).