Exercise Sheet 6

Please hand in your solutions by April 9, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. A parallelizable manifold M is one where the tangent bundle TM is isomorphic to the trivial bundle $M \times \mathbb{R}^m$. In other words, M carries m vector fields that are linearly independent at every point of M.

a) Prove that S^3 is parallelizable.

b) Prove that S^7 is parallelizable.

c) Prove that S^{2n} is not parallelizable for all $n \geq 1$.

Hint: For a) and b), see S^3, S^7 as the unit quaternions resp. unit octonions. It is a hard theorem that S^0, S^1, S^3 and S^7 are the only parallelizable spheres.

2. Let M be a compact manifold.

a) Prove that there exist a finite collection $X_1, X_2, \dots, X_k \in \mathrm{Vect}(M)$ such that $\{X_1(p), X_2(p), \cdots, X_k(p)\}$ spans T_pM at every $p \in M$.

b) Prove that the number k from $a)$ can be chosen such that $k \leq 2m$, where $m = \dim(M)$.

Hint: For b): Start with any large number of vector fields $X_1, \dots, X_k$ and consider the map $F : T_pM \to \mathbb{R}^k$ whose j-th coordinate is given by $F_j(p, v) := \langle v, X_j(p)\rangle$. What does Sard's theorem tell us when $k > 2m$ and how can you use this to construct $k-1$ vector fields $Y_1, \dots, Y_{k-1}$ which still span T_pM at every $p \in M$?

3. Let M be a connected manifold without boundary and let $p, q \in M$. Let $K \subset M$ be a compact set containing $p, q \in K^\circ$ in its non empty, connected interior. Then there exists an isotopy $\psi_t : M \to M$ for $t \in [0, 1]$ such that $\psi_0 = \mathrm{id}$, $\psi_1(p) = q$ and

$$\mathrm{supp}(\{\psi_t\}_{t\in[0,1]}) := \overline{\bigcup_{0\leq t\leq 1} \{x \in M \,:\, \psi_t(x) \neq x\}} \subset K.$$

We call this an isotopy with compact support in K.

Hint: This is a slight generalization of the homogenisation lemma from the lecture. Have a careful look at the proof and check how it can be adopted to encompass this case.

4. Let M^m be a compact, connected manifold without boundary where $m \geq 2$.

a) Let $\varphi : U \to \mathbb{R}^m$ be a chart defined on $U \subset M$ with $B_1(0) \subset \varphi(U)$. Prove that there exists $X \in \mathrm{Vect}(M)$ which does not vanish outside of $\varphi^{-1}(B_1(0))$ and has only isolated non-degenerate zeros in $\varphi^{-1}(B_1(0))$.

b) Let $Y : \overline{B_1(0)} \to \mathbb{R}^m$ be a vector field with isolated non-degenerate zeros. Suppose that $Y(x) \neq 0$ for $x \in \partial B_1(0)$ and $\sum_{x \in Y^{-1}(0)} \iota(Y, p) = 0$. Then there exists a nowhere vanishing vector field $Z : \overline{B_1(0)} \to \mathbb{R}^m$ which agrees with Y near the boundary.

c) Prove that if $\chi(M) = 0$ then there is a vector field $X \in \mathrm{Vect}(M)$ with no zeroes.

Hint: For a): Start with any vector field X with isolated non-degenerate zeros. Use the homogeneity lemma of the previous exercise to move all the zeros into a ball inside a coordinate chart of X.
For b): The Gauss map $\partial B_1(0) \to S^{m-1}$ defined by $x \mapsto Y(x)/\|Y(x)\|$ has degree zero under the given assumptions. Hence it is homotopic to a constant map by the Hopf degree theorem.
For c): Combine a) and b) and make sure that all your modifications produce a smooth vector field.

5. Let $f : M \to \mathbb{R}$ be a Morse function as in Exercise 2, Sheet 3. The **Morse Lemma** asserts that for every $p \in \mathrm{Crit}(f)$, there is a chart $\varphi_p : U_p \subset M \to \mathbb{R}^m$ with $\varphi_p(p) = 0$ such that

$$f \circ \varphi_p^{-1}(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2.$$

Here $k := \mu(f, p)$ is determined as the dimension of the negative eigenspace of the Hessian $H_p f$ and it is called the **Morse index** of f at p.

Let g be a Riemannian metric on M. The gradient vector field $X = \nabla_g f$ of a smooth function $f : M \to \mathbb{R}$ is defined by

$$g(X(p), \hat{p}) = df(p)\hat{p}, \qquad \text{for all } p \in M \text{ and } \hat{p} \in T_p M.$$

Prove the following properties for the gradient field of a Morse function f.

a) For every metric g the gradiant $X = \nabla_g f$ is a vector field with isolated, non-degenerate zeroes.

b) For every $p \in \mathrm{Crit}(f)$ and any two Riemannian metrics g_1, g_2 the indices $\iota(\nabla_{g_0} f, p) = \iota(\nabla_{g_1} f, p)$ agree.

c) For every $p \in \mathrm{Crit}(f)$ it holds $\iota(\nabla_g f, p) = (-1)^{\mu(f,p)}$.

6. **a)** Show that $f_n : \mathbb{R}P^n \to \mathbb{R}$ defined by

$$f_n([x_0 : x_1 : \ldots : x_n]) := \frac{\sum_{j=1}^n j x_j^2}{\sum_{j=0}^n x_j^2}$$

is a Morse function on $\mathbb{R}P^n$. Determine all the critical points of f_n and their Morse index.

b) Show that $\chi(\mathbb{R}P^n) = 0$ when n is odd and $\chi(\mathbb{R}P^n) = 1$ when n is even.

c) Show that $g_n : \mathbb{C}P^n \to \mathbb{R}$ defined by

$$g_n([z_0 : z_1 : \ldots : z_n]) := \frac{\sum_{j=1}^n j |z_j|^2}{\sum_{j=0}^n |z_j|^2}.$$

is a Morse function on $\mathbb{C}P^n$. Determine all the critical points of g_n and their Morse index.

d) Show that $\chi(\mathbb{C}P^n) = n + 1$ for $n \geq 1$.

Hint: Use the gradient vector fields of the Morse functions to compute the Euler characteristic in part b) and d) (see Exercise 5 above)