Exercise Sheet 7

Please hand in your solutions by April 16, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.
Announcement: The lecture notes for the course are being updated and now include a section on Intersection Theory.

1. **a)** For M^{2n-1} compact, oriented manifold without boundary, prove that $\chi(M) = 0$.

b) For $P, Q \subset N$ be two compact oriented submanifolds without boundary of complimentary dimension in an oriented manifold N. Prove that

$$I(Q, P) = (-1)^{\dim(Q)\dim(P)} I(P, Q).$$

c) Let n be odd. Suppose $Q^n \subset N^{2n}$ is a compact oriented submanifold in an oriented manifold N. Show that $I(Q, Q) = 0$.

d) For $N = \mathbb{R}P^2$ and $Q = \mathbb{R}P^1 \subset N$ prove that $I_2(Q, Q) = 1$. Why does this result not contradict c) ?

Hint: For a): Calculate the Euler characteristic using X and $-X$.

2. Let M be a compact, oriented manifold without boundary.

a) Let $\Delta = \{(x, x) : x \in M\}$ be the diagonal in $N = M \times M$. Prove that

$$\Delta \cdot \Delta = \chi(M). \tag{1}$$

b) The right-hand side of (1) does not depend on the orientation of M. Does a similar equation to (1) hold for non-oriented manifolds M?

c) Take the zero section $Q = \{(p, v) \in TM : v = 0\}$ in $N = TM$ and prove that

$$Q \cdot Q = \chi(M).$$

3. Let M^m, N^n, Q^{n-m} be compact, oriented manifolds without boundary. Let $f : M \to N$ and $g : Q \to N$ be smooth maps. Explain what it mean that $f \pitchfork g$ and define the intersection number $I(f, g)$. Let $\Delta \subset N \times N$ denote the diagonal. Then

a) $f \pitchfork g$ if and only if $f \times g \pitchfork \triangle$.

b) Show that $I(f, g) = (-1)^{\dim(Q)} I(f \times g, \triangle)$.

4. **a)** Let $f : T^2 \to T^2 : (x, y) \mapsto (ax + by, cx + dy)$ for $a, b, c, d \in \mathbb{Z}$. Prove that the degree and Lefschetz number are given by the formulae

$$\deg(f) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathrm{L}(f) = \det \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}.$$

b) Let $f : S^1 \to S^1$ be a smooth map. Then $\mathrm{L}(f) = 1 - \deg(f)$.

c) Let $f : S^2 \to S^2$ be a smooth map. Then $\mathrm{L}(f) = 1 + \deg(f)$.

Hint: For a), to calculate the degree, set $f_A : T^2 \to T^2 : x \to Ax$. Notice that $\deg(f_{AB}) = \deg(f_A)\deg(f_B)$. With this, you can reduce the general case for $A \neq 0$ to the case $a > 0$ and $\det A \geq 0$. After that, use the identity

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} A \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ ac & a \det A \end{pmatrix}.$$

The Lefschetz Number can be deduced from the degree formula. For b) and c), Hopf degree lemma comes in handily. For c), you can use $\mathbb{C}P^1 \cong S^2$ and use exercise 5 from sheet 4. For maps of degree -1, think antipodal map.

5. **a)** For $n < m$ prove that $I_2(\mathbb{R}P^m, \mathbb{R}P^{n-m}) = 1$ in $\mathbb{R}P^n$.

b) Prove that $\mathbb{R}P^n$ is not simply connected.

6. Let $f : \mathbb{C} \to \mathbb{C}^2$ be a holomorphic function with coordinates $f(z) := (u(z), v(z))$ and assume that u is non-constant. Let $Q := \{0\} \times \mathbb{C}$.

a) Prove that the intersections of f with Q are all isolated.

b) Prove that the local intersection indices $\iota(z_0; f, Q)$ are all positive.

7. Let $Q \subset N$ be a compact submanifold without boundary in a Riemannian manifold N. Prove that there exists a normal vector field $X \in \mathrm{Vect}^\perp(Q)$ such that every zero of X is non-degenerate.

Hint: Take a tubular neighbourhood $V_\epsilon := \{\exp_p(v) : q \in Q, v \in (T_qQ)^\perp, |v| < \epsilon\}$ of Q. Thom-Smale transversality shows that the exists a embedding map $f_1 : Q \to N$ which is homotopic to the inclusion $f_0 = \iota_Q$ and transverse to Q. Why can we assume that $f_1(Q) \subset V_\epsilon$? (Have a careful look at the proof)

8. * Let $Q^n \subset N^{2n}$ be two compact oriented manifolds without boundary and assume that $Q \cdot Q = 0$. Prove that there exists a diffeomorphism $\varphi : N \to N$ isotopic to the identity such that $\varphi(Q) \cap Q = \emptyset$.

Hint: First, construct a normal vector field $X \in \mathrm{Vect}(Q)^\perp$ with no zeros. This uses similar ideas as Exercise 4 on Sheet 6: Start with a vector field X as in Exercise 7, then use the homogeneity Lemma to transport all zeros into a single chart: To transport X along an isotopy $\{\phi_t\}_{0 \leq t \leq 1}$ use parallel transport and define $X_1(q) = \Phi^\perp_{\gamma_q}(0,1)X(q)$ where $\gamma_q : [0,1] \to Q$ is the path $\gamma(t)\phi_t(q)$. Finally modify the normal vector field in a chart to obtain a nowhere vanishing normal vector field.