Exercise Sheet 8

Please hand in your solutions by April 23, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. **a)** Let $\dim(V) = n$ and let $\alpha_i \in V^*$, $v_i \in V$ for $i = 1, \ldots, n$. Prove that

$$(\alpha_1 \wedge \ldots \wedge \alpha_n)(v_1, \ldots, v_n) = \det(\alpha_j(v_j)_{ij}).$$

b) Prove that $dx^I = dx^{i_1} \wedge \ldots dx^{i_k}$ for $I = (i_1, \ldots, i_k) \in \mathcal{I}_k$.

c) Let $\dim(V) = n$. Prove that for a linear map $\Phi : V \to V$ and $\omega \in \Lambda^n V^*$, we have

$$\Phi^* \omega = \det(\Phi)\omega.$$

2. We look at the exterior differential on $\mathbb{R}^3$.

a) For $f \in \Omega^0(\mathbb{R}^3) = C^\infty(\mathbb{R}, \mathbb{R}^3)$, prove that

$$df = \sum_{i=1}^{3} \mathbf{grad}(f)_i \, dx^i,$$

where $\mathbf{grad}(f) = (\partial_1 f, \partial_2 f, \partial_3 f)$.

b) For $\alpha \in \Omega^1(\mathbb{R}^3)$ with $\alpha = g_1 dx^1 + g_2 dx^2 + g_3 dx^3$, prove that

$$d\alpha = \mathbf{curl}(g)_1 \, dx^2 \wedge dx^3 + \mathbf{curl}(g)_2 \, dx^3 \wedge dx^1 + \mathbf{curl}(g)_3 \, dx^1 \wedge dx^2,$$

where $\mathbf{curl}(g) = (\partial_2 g_3 - \partial_3 g_2, \partial_3 g_1 - \partial_1 g_3, \partial_1 g_2 - \partial_2 g_1)$ for $g \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

c) For $\omega \in \Omega^2(\mathbb{R}^3)$ with $\omega = h_1 dx^2 \wedge dx^3 + h_2 dx^3 \wedge dx^1 + h_3 dx^1 \wedge dx^2$, prove that

$$d\omega = \mathbf{div}(h) \, dx^1 \wedge dx^2 \wedge dx^3,$$

where $\mathbf{div}(h) = \partial_1 h_1 + \partial_2 h_2 + \partial_3 h_3$ for $h \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

d) Prove that $\mathbf{curl} \circ \mathbf{grad} = 0$ and $\mathbf{div} \circ \mathbf{curl} = 0$.

So we have

$$\Omega^0(\mathbb{R}^3) \xrightarrow{d=grad} \Omega^1(\mathbb{R}^3) \xrightarrow{d=cur\ell} \Omega^2(\mathbb{R}^3) \xrightarrow{d=div} \Omega^3(\mathbb{R}^3).$$

3. Prove that a compact manifold M^m is orientable if and only if there exists a nowhere vanishing m-form ω on M. Such a form is called **volume form**.

4. Let V be a m-dimensional real vector space and $\omega \in \Lambda^k V^* \setminus \{0\}$ be an alternating k-form on V. The kernel of ω is the linear subspace

$$\ker(\omega) := \{v \in V \, : \, \omega(v, v_2, \ldots, v_k) = 0 \text{ for all } v_2, \ldots, v_k \in V\}.$$

a) Prove that $\dim(\ker(\omega)) \leq m - k$.

b) We call $\omega \in \Lambda^k V^*$ *decomposable* if there are $\alpha_i \in V^*$ such that

$$\omega = \alpha_1 \wedge \ldots \wedge \alpha_k.$$

Show that ω is decomposable if and only if $\dim(\ker(\omega)) = m - k$.

c) Show that for $\dim(V) < 4$, every k-form is decomposable.

d) We call $\omega \in \Lambda^k V^*$ *non-degenerate* if $\ker(\omega) = 0$. Find an example of a non-degenerate 2-form on $\mathbb{R}^4$.

5. The **Hodge star** operator $* \; : \Omega^k(\mathbb{R}^m) \to \Omega^{m-k}(\mathbb{R}^m)$ is the unique map satisfying

$$\eta \wedge *\omega = \langle \omega, \eta \rangle dx^1 \wedge \cdots \wedge dx^m$$

for all $\omega, \eta \in \Omega^k(\mathbb{R}^m)$. Here the inner product of two k-forms is defined by $\left\langle \sum_{I \in \mathcal{I}_k} a_I dx^I, \sum_{I \in \mathcal{I}_k} b_I dx^I \right\rangle = \sum_{I \in \mathcal{I}_k} a_I b_I$.

a) Calculate $*\omega \in \Omega^1(\mathbb{R}^3)$ for

$$\omega = a_{12} dx^1 \wedge dx^2 + a_{13} dx^1 \wedge dx^3 + a_{23} dx^2 \wedge dx^3 \in \Omega^2(\mathbb{R}^3)$$

b) Show that $* \; : \Omega^k(\mathbb{R}^m) \to \Omega^{m-k}(\mathbb{R}^m)$ is linear.

c) Calculate all $\omega \in \Omega^2(\mathbb{R}^4)$ with $\omega = *\omega$.

d) Calculate $* * \omega$ for a general form $\omega \in \Omega^k(\mathbb{R}^m)$.

6. **a)** Define $\omega \in \Omega^2(S^2)$ by

$$\omega_x = x_1 dx^2 \wedge dx^3 + x_2 dx^3 \wedge dx^1 + x_3 dx^1 \wedge dx^2.$$

Use Stokes' theorem to establish a relation between $\text{area}(S^2) := \int_{S^2} \omega$ and $\text{vol}(B_1(0))$. Calculate either one of them, to deduce the other.

b) Define $\rho \in \Omega^n(S^n)$ by

$$\rho_x := \sum_{i=1}^{n+1} (-1)^{j-1} x_j dx^1 \ldots dx^{j-1} \wedge \widehat{dx^j} \wedge dx^{j+1} \ldots dx^{n+1},$$

This is obtained by plugging x into the first coordinate of the standard volume form on $\mathbb{R}^{n+1}$. Prove that ρ is invariant under $\mathrm{SO}(n+1)$ and that ρ is a volume form on S^n.