Exercise Sheet 9

Please hand in your solutions by April 30, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. **a)** If $f : M \to N$ is an orientation preserving diffeomorphism between oriented m-manifolds then $\int_M f^*\omega = \int_N \omega$ for every $\omega \in \Omega_c^m(N)$. If $f : M \to N$ is an orientation reversing diffeomorphism between oriented m-manifolds then $\int_M f^*\omega = -\int_N \omega$ for every $\omega \in \Omega_c^m(N)$.

b) The orientation double cover of a manifold M is

$$\tilde{M} := \{(p, o) \,:\, p \in M,\, o \text{ is an orientation of } T_pM\}.$$

Show that $\tilde{M}$ has a natural smooth manifold structure and is orientable. Moreover, interchaning the orientation yields an orientation reversing a diffeomorphism $\varphi : \tilde{M} \to \tilde{M}$ with $\varphi^2 = \mathrm{id}_{\tilde{M}}$.

c) Let $\pi : \tilde{M} \to M$ denote the canonical projection. Then $\tilde{\eta} \in \Omega^k(\tilde{M})$ satsifies $\tilde{\eta} = \varphi^*\tilde{\eta}$ if and only if there exists $\eta \in \Omega^k(M)$ with $\tilde{\eta} = \pi^*\eta$.

d) Let M be a compact connected non-orientable m-manifold without boundary. Prove that every m-form on M is exact.

2. Let M, N be smooth manifolds.

a) Prove that given $\varphi : M \to N$ a diffeomorphism, $Y \in \mathrm{Vect}(N)$, and ω a form on M, we have

$$\mathcal{L}_{\varphi^* Y}(\varphi^*\omega) = \varphi^*(\mathcal{L}_Y \omega).$$

b) Let $Y_t \in \mathrm{Vect}(N)$ be a smooth family of vector fields and let ψ_t be the isotopy generated by Y_t via

$$\partial_t \psi_t = Y_t \circ \psi_t,\ \psi_0 = \mathrm{id}.$$

Prove that

$$\frac{d}{dt}\psi_t^*\omega = \psi_t^*\mathcal{L}_{Y_t}\omega.$$

c) Let $\omega \in \Omega^k(N)$, $Y \in \mathrm{Vect}(N)$ and $\varphi : [0,1] \times M \to N : (t,p) \mapsto \varphi_t(p)$ a smooth map. Deduce from

$$\mathcal{L}_Y \omega = d(\iota(Y)\omega) + \iota(Y)d\omega, \tag{1}$$

the formula

$$\frac{d}{dt}\varphi_t^*\omega = dh_t\omega + h_t d\omega \tag{2}$$

where we recall $h_t : \Omega^k(N) \to \Omega^{k-1}(M)$ is given by

$$(h_t\omega)_p(v_1, \ldots, v_{k-1}) := \omega_{\varphi_t(p)}(\partial_t\varphi_t(p), d\varphi_t(p)v_1, \ldots, d\varphi_t(p)v_{k-1})$$

for $p \in M$ and $v_1, \ldots, v_{k-1} \in T_pM$.

Hint: Assume first that the map $\varphi_t : M \to N$ is an embedding for every t. Then there is a smooth family of vector field such that

$$Y_t \in \mathrm{Vect}(N),\ Y_t \circ \varphi_t = \partial_t \varphi_t.$$

Let ψ_t be isotopy of N generated by Y_t as above. Then $\varphi_t = \psi_t \circ \varphi_0$. Now deduce (2) from (1) for $\mathcal{L}_{Y_t}\omega$. To prove (2) in general replace the map $\varphi_t : M \to N$ by the embedding $\psi_t : M \to \tilde{N} := M \times N, \tilde{\psi}_t(p) := (p, \varphi_t(p))$ and argue as above.

3. Let M be a manifold, $X \in \mathrm{Vect}(M)$ and τ, ω be forms on M. Prove the following:

a) $\iota(X)(\omega \wedge \tau) = (\iota(X)\omega) \wedge \tau + (-1)^{\deg(\omega)}\omega \wedge (\iota(X)\tau)$.

b) $\mathcal{L}_X(\omega \wedge \tau) = \mathcal{L}_X\omega \wedge \tau + \omega \wedge \mathcal{L}_X\tau$.

4. Prove that for $\beta \in \Omega^1(M)$, $\omega \in \Omega^2(M)$, and $X, Y, Z \in \mathrm{Vect}(M)$, we have

a) $d\beta(X,Y) = \mathcal{L}_X(\beta(Y)) - \mathcal{L}_Y(\beta(X)) + \beta([X,Y])$,

b) $$\begin{aligned} d\omega(X,Y,Z) = {} & \mathcal{L}_X(\omega(Y,Z)) + \mathcal{L}_Y(\omega(Z,X)) + \mathcal{L}_Z(\omega(X,Y)) \\ & + \omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y). \end{aligned}$$

5. Let M be a simply connected compact m-manifold.

a) Prove that any closed 1-form is exact.

b) Prove that any map $M \to T^m$ has degree zero.

Hint: For b), choose a product of 1-forms as volume form on T^m.

6. Let M be a smooth compact connected oriented m-manifold without boundary and denote by

$$\mathcal{V}(M) := \left\{ \omega \in \Omega^m(M) \,:\, \omega \text{ is a volume form with } \int_M \omega = 1. \right\}$$

the space of volume forms with volume 1. Fix throughout the exercise $\omega_0 \in \mathcal{V}(M)$ and define

$$\mathrm{Diff}(M, \omega_0) := \{ \varphi \in \mathrm{Diff}(M) \,:\, \varphi^*\omega_0 = \omega_0 \}$$

The goal of this exercise is to show that the inclusion of $\mathrm{Diff}(M, \omega_0)$ into the space of orientation-preserving diffeomorphisms $\mathrm{Diff}^+(M)$ is a homotopy equivalence.

a) Prove that for every $\omega \in \mathcal{V}(M)$ the linear map

$$I_\omega : \mathrm{Vect}(M) \to \Omega^{m-1}(M), \quad \text{given by } I_{\omega_0}(X) := \iota(X)\omega$$

is an isomorphism.

b) * Show that there exists a continuous map

$$\mathcal{V}(M) \to \mathrm{Diff}^+(M), \qquad \omega \mapsto \psi_\omega$$

such that $\psi_{\omega_0} = \mathrm{id}_M$ and $\psi_\omega^*\omega = \omega_0$ for every $\omega \in \mathcal{V}(M)$.

c) * Show that the map

$$\mathrm{Diff}^+(M) \to \mathcal{V}(M) \times \mathrm{Diff}(M, \omega_0), \qquad \psi \mapsto (\psi^*\omega_0, \psi \circ \psi_{\psi^*\omega_0})$$

is a well-defined homeomorphism with inverse $(\omega, \phi) \mapsto \phi \circ \psi_\omega^{-1}$.

d) Conclude that the inclusion of $\mathrm{Diff}(M, \omega_0)$ into $\mathrm{Diff}^+(M)$ is a homotopy equivalence.