# Exercise Sheet 10

Please hand in your solutions by May 7, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

**1.** Prove that a geodesically convex subset $U \subset (M, g)$ is contractible.

**2.** Show that $\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\})$ is closed 1-form, but it is not exact. What does this tells us about the de-Rham complex of $\mathbb{R}^2 \setminus \{(0,0)\}$?

**3.** Let $\alpha \in \Omega^1(M)$. Equivalent are

**a)** $\alpha$ is exact.

**b)** For any two smooth curves $\gamma_0, \gamma_1 : [0,1] \to M$ which agree at the end, we have

$$\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha.$$

**c)** For any smooth loop $\gamma : S^1 \to M$, we have

$$\int_{\gamma} \alpha = 0.$$

**4.** Let $\alpha \in \Omega^1(M)$. Equivalent are

**a)** $\alpha$ is closed.

**b)** For any two smooth curves $\gamma_0, \gamma_1 : [0,1] \to M$ homotopic with fixed end points, we have

$$\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha.$$

**c)** For any contractible, smooth loop $\gamma : S^1 \to M$, we have

$$\int_{\gamma} \alpha = 0.$$

**5.** Let $M$ be a compact manifold with non-empty boundary.

**a)** Prove that there is $X \in \mathrm{Vect}(M)$ such that $X$ points in on the boundary.

**b)** Prove that $X$ as in a) has a smooth semi-flow

$$\varphi : [0, \infty) \times M \to M : (t, p) \mapsto \varphi_t(p).$$

**c)** Every map $f : M \to M$ is homotopic to a map without fixed points on the boundary.

**d)** Let $f_0, f_1 : M \to M$ be homotopic maps without fixed points on the boundary. Prove there is a homotopy $(f_t)_{t\in[0,1]}$ such that $f_t(q) \neq q$ for all $t \in [0, 1]$ and all $q \in \partial M$.

**e)** Define the Lefschetz number $L(f)$ for $f : M \to M$ without fixed points on the boundary.

**f)** Prove that there is an open neighbourhood $U \subset M$ of the boundary $\partial M$ such that $[0, 1) \times \partial M \cong U$.