Exercise Sheet 11

Please hand in your solutions by May 14, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. Let M be a smooth m-manifold (with or without boundary). Let $U, V \subset M$ be open sets such that $M = U \cup V$.

a) Let ρ_U, ρ_V be a partition of unity subordinate to the U, V. Let $\omega \in \Omega^k(U \cap V)$ and define $d^* : \Omega^k(U \cap V) \to \Omega^{k+1}(M)$ by

$$d^*\omega = \begin{cases} d\rho_U \wedge \omega & \text{on } U \cap V \\ 0 & \text{on } M \setminus (U \cap V). \end{cases}$$

Find an example of M, U, V and $\omega \in \Omega^k(U \cap V)$ such that $d^*\omega$ is not exact.

b) Prove that the cohomology class $[d^*\omega] \in H^{k+1}(M)$ is independent of the choice of partition of unity.

Hint: For *a*) take for example $M = S^1$ and $k = 0$.

2. **a)** Calculate the dimensions of the de Rham cohomology groups of $M \times S^1$.

b) Calculate the dimensions of the de Rham cohomology groups of the torus T^m.

Hint: For a): Use the Künneth formula or Mayer–Vietoris: For the later approach cover S^1 by two open intervals to obtain an open cover $M \times S^1 = U \cup V$. Have a careful look at the maps $H^k(U) \oplus H^k(V) \to H^k(U \cap V)$. What can you say about its kernel and image?

3. Calculate the dimensions of the de Rham cohomology groups for the following manifolds.

a) $\mathbb{R}P^2$ by using the double covering $S^2 \to \mathbb{R}P^2$.

b) $\mathbb{R}P^n$ for $n \geq 2$.

c) $\mathbb{C}P^n$ for $n \geq 1$ by induction.

4. Prove the five-lemma stated below.

Let $A_i, B_i,\ i = 1, 2, 3, 4, 5$ be abelian groups. Let

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
\end{array}$$

be a commutative diagram in which the horizontal sequences are exact. If h_1, h_2, h_4 and h_5 are isomorphisms prove that h_3 is also an isomorphism.

Hint: Chasing diagrams is fun.

5. The goal is to prove the following.

Every non-empty geodesically convex open subset of a Riemannian m-manifold M without boundary is diffeomorphic to $\mathbb{R}^m$.

We argue in several steps.

a) Prove or assume that it is diffeomorphic to a bounded **star shaped** open set $U \subset \mathbb{R}^m$ centred at the origin, so that if $x \in U$, then $tx \in U$ for $0 \leq t \leq 1$.

b) Prove that there exists a smooth function $g : \mathbb{R}^m \to \mathbb{R}$ such that $g(x) > 0$ for every $x \in U$, $g(x) = 1$ for $|x|$ sufficiently small, and $g(x) = 0$ for $x \in \mathbb{R}^m \setminus U$.

c) Given $g : \mathbb{R}^m \to \mathbb{R}$ as in b). Define $h : U \to [0, \infty)$ by

$$h(x) := \int_0^1 \frac{dt}{g(tx)}.$$

Prove that the map $\varphi : U \to \mathbb{R}^m, \varphi(x) := h(x)x$, is a diffeomorphism.

Hint: For b), use the fact that U is second-countable and take a countable cover by balls of small radius $r_i \leq 1$. Fix one cut off function ρ which is > 0 on $B_1(0)$ and 0 outside of $B_1(0)$. The constants C_i bounding all partial derivatives up to order i can come in handy. You can define g (dropping the condition of constant near 0 for now) through an infinite sum where summands involve ρ, C_i, r_i and some weighting. Use Weierstrass M-test a lot.

Note: There are contractible manifolds without boundary which are not diffeomorphic to $\mathbb{R}^m$, e.g. an exotic $\mathbb{R}^4$.

6. Prove that for M compact, $M \setminus \partial M$ has a finite good cover.

Hint: Use small geodesically convex balls and Exercise 5. Small can mean of radius less than one fourth the injectivity radius of M for $\partial M = \emptyset$.