Exercise Sheet 12

Please hand in your solutions by May 21, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. Let M be a compact manifold M without boundary and consider the map

$$H^1(M) \to \mathrm{Hom}(\pi_1(M), \mathbb{R}) : \ [\alpha] \mapsto \left([\gamma] \mapsto \int_\gamma \alpha\right)$$

a) Show that the map is well-defined.

b) Show that the map is injective.

c) ** Show that the map is surjective.

Hint: Use Exercise 3 on Sheet 10 to prove injectivity of the map. Surjectivity is the hard part and we did not succeed in finding any rigorous solution, which works without assuming some non-trivial facts from algebraic geometry and the de Rham theorem. There is a convincing heuristic which uses the universal cover $\tilde{M} \to M$ and $H^1(\tilde{M}) = 0$ to construct a preimage.

2. Let M_0 and M_1 be smooth, closed and oriented m-manifolds. Let

$$\iota_0 : D^m \to M_0, \qquad \iota_1 : D^m \to M_1$$

be two embeddings of the unit ball $D^m \subset \mathbb{R}^m$, where ι_0 is orientation preserving and ι_1 is orientation reversing. The connected sum $M_0 \# M_1$ is defined as

$$M_0 \# M_1 := \left((M_0 \setminus \{\iota_0(0)\}) \,\dot\cup\, (M_1 \setminus \{\iota_1(0)\}) \right) / \sim$$

where $\dot\cup$ denotes the disjoint union and $\iota_0(tx) \sim \iota_1((1-t)x)$ for $x \in S^{m-1}$ and $0 < t < 1$.

a) Let M be a smooth closed and oriented m-manifold and let $p \in M$. Compute the de Rham cohomology of $M \setminus \{p\}$ using the Mayer–Vietoris sequence

b) Compute the de Rham cohomology $H^k(M_0 \# M_1)$ using the Mayer-Vietoris sequence.

c) Calculate the de Rham cohomology of the connected sum $\Sigma_\ell := T^2 \# \ldots \# T^2$ of $\ell \geq 2$ copies of the two torus.

3. **a)** Specify a basis of $H^1(T^2)$ and compute the Poincaré pairing on this basis.

b) Prove that both $\dim H^1(\Sigma)$ and the Euler characteristic $\chi(\Sigma)$ is even for every oriented, compact 2-manifold Σ without boundary, a so-called Riemann surface.

c) We call $g := \frac{1}{2}\dim H^1(\Sigma)$ the genus of a Riemann surface Σ. Give an example of a Riemann surface Σ_g for every $g \in \mathbb{N}$.

Hint: For b), use Poincaré duality and the expression of the Euler characteristic in terms of Betti numbers.

Note: The classification of Riemann surfaces proves that all Riemann surfaces of the same genus are diffeomorphic.

4. **a)** Give an example of a smooth map $f : M \to N$ between manifolds without boundary and a compactly supported form $\omega \in \Omega_c^k(N)$ such that $f^*\omega$ is not compactly supported.

b) Let M be the open Möbius strip. Compute the de-Rham cohomology groups – the compactly supported and also the full one.

c) Give a counter-example to Poincaré duality in the non-oriented case.

5. [**Poincaré lemma**] Find an explicit formula for an operator

$$h : \Omega^k(\mathbb{R}^m) \to \Omega^{k-1}(\mathbb{R}^m)$$

that satisfies $d \circ h + h \circ d = \mathrm{Id}$.

Hint: Look carefully at the proof of Theorem 1 in Chapter XI. Use the homotopy $f_t(x) = tx$ and Cartan's formula.

6. Let M be a compact manifold without boundary and $U \subset M$ be an open set. Assume that $f : M \to M$ has $f(M) \subset U$. Prove that for every $k \in \mathbb{N}$,

$$\mathrm{tr}\left(f^* : H^k(M) \to H^k(M)\right) = \mathrm{tr}\left((f|_U)^* : H^k(U) \to H^k(U)\right).$$

Hint: We use the Mayer-Vietoris sequence with U and $V = M \setminus f(M)$. Complete the following diagram such that the resulting diagram commutes.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^{k-1}(U\cap V) & \xrightarrow{d^*} & H^k(M) & \xrightarrow{i^*} & H^k(U)\oplus H^k(V) & \xrightarrow{j^*} & H^k(U\cap V) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f^*} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^{k-1}(U\cap V) & \xrightarrow{d^*} & H^k(M) & \xrightarrow{i^*} & H^k(U)\oplus H^k(V) & \xrightarrow{j^*} & H^k(U\cap V) & \longrightarrow & \cdots
\end{array}$$