Exercise Sheet 13

Please hand in your solutions by May 28, 2018. If you have any troubles with understanding the material of the lecture or solving the exercises, please ask questions in your exercise class.

1. **a)** Let $E \to M$ be a vector bundle. Show that for every $p \in M$ and for every $e \in E_p$ there exists a smooth section s such that $s(p) = e$.

b) For E a vector bundle. We introduce two notions of orientatability.

(i) E has transition maps in $\mathrm{GL}_+(\mathbb{R}^n)$.

(ii) There exists a collection of orientations of the fibres of E, such that for every $p_0 \in M$ there exists an open neighbourhood of $U \subset M$ of p_0 and there exist smooth sections $s_1, s_2, \cdots, s_n : U \to E|_U$ such that $s_1(p), s_2(p), \cdots, s_n(p)$ form a positive basis of E_p for all $p \in U$.

Prove that these two definitions are equivalent.

c) Assume that M is an oriented manifold and $E \to M$ is a vector bundle. Show that E is oriented as a vector bundle if and only if E is oriented as a manifold.

2. Let M be a compact manifold and let $\pi^E : E \to M$ and $\pi^F : F \to M$ be vector bundles. A vector bundle homomorphism is a smooth map $\Phi : E \to F$ such that $\pi^F \circ \Phi = \pi^E$ and $\Phi|_{E_p} : E_p \to F_p$ is a linear map for every p.

a) If Φ is injective prove that Φ is an embedding.

b) If Φ is bijective prove that Φ is a diffeomorphism.

c) Prove that for every vector bundle E over M, there exists an injective vector bundle homomorphism $\Phi : E \to M \times \mathbb{R}^N$ for some N.

Hint: In a), for properness use the following two ingredients.

- On a manifolds, $f : M \to N$ is proper, iff any sequence (x_k) such that $f(x_k)$ is bounded, is itself bounded. Here boundedness is with respect to any metric inducing the manifold topology on M and N.
- For any smooth family $A(x) \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^\ell)$ of injective linear maps, there is a constant $C > 0$ such that $|v| \leq C|A(x)v|$ for $v \in \mathbb{R}^m$ and any x on a small neighbourhood. For this construct smooth families of left inverses to $A(x)$.

3. **a)** Let $Q \subset (M, g)$ be a submanifold. Prove that the normal bundle $TQ^{\perp_g}$ is a vector bundle over Q.

b) Let $E, F \to M$ be two vector bundles. Prove that the Whitney sum

$$E \oplus F := \{(p, e, f) : p \in M, e \in E, f \in F, \pi^E(e) = \pi^F(f) = p\}$$

is a vector bundle over M.

c) Let $E, F \to M$ be two vector bundles. Prove that the homomorphism bundle

$$\mathrm{Hom}(E,F) := \{(p,\Phi) : p \in M,\ \Phi : E_p \to F_p \text{ is linear}\}$$

is a vector bundle over M.

Hint: For a), use the Gram-Schmidt method on a suitably chosen basis of vector fields around any point and see that smoothness is preserved.

4. Let E be a real rank-n vector bundle over a smooth m-manifold M and let $s : M \to E$ be a smooth section of E. Assume s is transverse to the zero section. Then the zero set

$$s^{-1}(0) := \{p \in M : s(p) = 0_p\}$$

of s is a smooth submanifold of M of dimension $m - n$ and

$$T_p s^{-1}(0) = \ker Ds(p)$$

for every $p \in M$ with $s(p) = 0_p$.

5. In this exercise, we introduce some important line bundles over $\mathbb{C}P^n$. Define for $d \in \mathbb{Z}$ the quotient

$$H^d := \Big((\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}\Big) \Big/ \sim$$

where $(z_0, \ldots, z_n; \zeta) \sim (\lambda z_0, \ldots, \lambda z_n; \lambda^d \zeta)$.

a) Show that H^d is a real 2-dimensional vector bundle over $\mathbb{C}P^n$. Find explicit trivializations of this bundle and compute the corresponding transition maps.

b) Find an isomorphism between H^{-1} and the tautological line bundle

$$E := \{(\ell, w) \in \mathbb{C}P^n \times \mathbb{C}^n \ :\ w \in \ell\}$$

c) Find an isomorphism between H^1 and the canonical line bundle

$$H := \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C}).$$

d) Show that $T\mathbb{C}P^n \oplus \mathbb{C} \cong H \oplus \ldots \oplus H$ with $(n+1)$ copies of H.

Hint: For d): It holds $T_\ell \mathbb{C}P^n \cong \mathrm{Hom}_C(\ell, \ell^\perp)$, see Exercise 6 on Sheet 1.

6. Let $E \to M$ be an oriented smooth vector bundle over and oriented compact m-manifold M without boundary. For $\omega \in \Omega^\ell(M)$ and $\tau \in \Omega_c^{n+k}(E)$ show that $\pi_*(\pi^*\omega \wedge \tau) = \omega \wedge \pi_*\tau$.