# Holiday Sheet 14

No model solutions will be provided and you should not hand in your solutions. Have a nice summer vacation.

**1.** The Grassmannian of $k$-planes in $\mathbb{R}^n$ is defined as the set

$$\mathrm{Gr}(n,k) := \{E \mid E \subset \mathbb{R}^n \text{ is a } k\text{-dimensional linear subspace}\}.$$

**a)** Show that as a set $\mathrm{Gr}(n,k)$ admits the following descriptions

(i) $\mathrm{Gr}(n,k) \cong \{A \in \mathbb{R}^{n\times k} \mid \det(A^T A) \neq 0\}/\mathrm{GL}(k,\mathbb{R})$

(ii) $\mathrm{Gr}(n,k) \cong \{A \in \mathbb{R}^{n\times k} \mid A^T A = \mathbb{1}\}/\mathrm{O}(k)$

where $\mathrm{GL}(k,\mathbb{R})$ and $\mathrm{O}(k)$ act by multiplication from the right side.

**b)** Endow $\mathrm{Gr}(n,k)$ with the structure of a smooth manifold. For this provide explicit charts and investigate the transition maps. In order to verify that you found the *right* atlas, verify that the projection map from $\{A \in \mathbb{R}^{n\times k} \mid \det(A^T A) \neq 0\}$ onto $\mathrm{Gr}(n,k)$ is a smooth submersion.

**c)** What is the dimension of $\mathrm{Gr}(n,k)$?

**d)** Show that there exists a canonical isomorphism

$$T_E\mathrm{Gr}(n,k) \cong \mathrm{Hom}(E,E^\perp).$$

**Hint:** This generalizes our discussion of projective space $\mathbb{R}P^n = \mathrm{Gr}(n+1,1)$. Revisit our discussion of these spaces if you get stuck.

**2.** The Grassmannian of complex $k$-planes in $\mathbb{C}^n$ is defined as the set

$$\mathrm{Gr}^{\mathbb{C}}(n,k) := \{E \mid E \subset \mathbb{C}^n \text{ is a complex } k\text{-dimensional linear subspace}\}.$$

**a)** Show that as a set $\mathrm{Gr}^{\mathbb{C}}(n,k)$ admits the following descriptions:

(i) $\mathrm{Gr}^{\mathbb{C}}(n,k) = \{A \in \mathbb{C}^{n\times k} \mid \det(A^* A) \neq 0\}/\mathrm{GL}(k,\mathbb{C})$

(ii) $\mathrm{Gr}^{\mathbb{C}}(n,k) = \{A \in \mathbb{C}^{n\times k} \mid A^* A = \mathbb{1}\}/\mathrm{U}(k)$

where $\mathrm{GL}(k,\mathbb{C})$ and $\mathrm{U}(k)$ act by multiplication from the right side.

**b)** Endow $\mathrm{Gr}^{\mathbb{C}}(n,k)$ with the structure of a smooth manifold. For this provide explicit charts and investigate the transition maps. In order to verify that you found the *right* atlas, verify that the projection map from $\{A \in \mathbb{C}^{n\times k}, \mid \det(A^* A) \neq 0\}$ onto $\mathrm{Gr}^{\mathbb{C}}(n,k)$ is a smooth submersion.

**c)** What is the (real) dimension of $\mathrm{Gr}^{\mathbb{C}}(n,k)$?

**d)** Show that there exists a canonical isomorphism

$$T_E\mathrm{Gr}^{\mathbb{C}}(n,k) \cong \mathrm{Hom}(E, E^{\perp}).$$

**Hint:** This generalizes our discussion of projective space $\mathbb{C}P^n = \mathrm{Gr}^{\mathbb{C}}(n+1,1)$. Revisit our discussion of these spaces if you get stuck.

**3.** The tautological vector bundle over the Grassmannian $\mathrm{Gr}(n,k)$ is defined as

$$E := \{(P,v) \,|\, P \in \mathrm{Gr}(n,k),\ v \in P\}$$

**a)** Show that $E$ defines a smooth rank $k$ vector bundle over $\mathrm{Gr}(n,k)$.

**b)** Let $M$ be a compact manifold and let $F$ be a smooth rank $k$ vector bundle vector $M$. Show that there exists a smooth map $f : M \to \mathrm{Gr}(n,k)$ into some Grassmannian such that

$$F = f^*E := \{(p,v) \,|\, p \in M,\ v \in E_{f(p)} = f(p)\}$$

where $E \to \mathrm{Gr}(n,k)$ denotes the tautological bundle.

**Remark:** We show in the next exercise that two homotopic maps give rise to isomorphic vector bundles. It then follows that one can classify vector bundles over $M$ in terms of homotopy classes of maps from $M$ to $\mathrm{Gr}(n,k)$.

**Hint:** For both parts, the extrinsic point of view from the last semester might be useful: Suppose $M \subset \mathbb{R}^N$ and $E \subset \mathbb{R}^N \times \mathbb{R}^n$ with linear subspaces in $\mathbb{R}^n$ as fibres. Then $E$ is a smooth vector bundle if and only if the associated field of orthogonal projections is smooth. Recall that every intrinsically defined vector bundle $E \to M$ admits such an embedding.

**4.** **a)** * Let $M$ and $N$ be smooth manifolds and let $E$ be a smooth vector bundle over $N$ and let

$$f : [0,1] \times M \to N, \qquad f(t,p) := f_t(p)$$

be a smooth homotopy. Choose a connection $\nabla$ on the bundle $f^*E$ over $[0,1] \times M$. Then parallel transport along the curves $\gamma_p(t) := (t,p)$ gives rise to a bundle isomorphism $f_0^*E \cong f_1^*E$. Fill in the necessary details of this argument!

**b)** Show that every vector bundle over a contractible space is trivial.

**Hint:** This exercise requires some background on connections on vector bundles and parallel transport which were not covered in the lecture. Part b) follows from the observation that the pullback bundle under a constant map is trivial.

**5.** **a)** Let $\Sigma$ be a closed oriented surface. Show that the bundle $T\Sigma \oplus \mathbb{R} \cong \Sigma \times \mathbb{R}^3$ is trivial.

**b)** Let $Y$ be a closed oriented 3-dimensional manifold. Show that for every 2-dimensional closed oriented submanifold $\Sigma \subset Y$ the restriction $TY|_\Sigma \cong \Sigma \times \mathbb{R}^3$ is trivial.

**c)** ** If you have the necessary background in algebraic topology, you can try to conclude that $TY \cong Y \times \mathbb{R}^3$ is trivial.

**Hint:** For c): First choose a triangulation of $Y$. Then $TY$ can be trivialized over the 1-skeleton, because $Y$ is orientable. Part b) asserts that the second Stiefel–Whitney class $w_2(TY) = 0$ vanishes and hence $TY$ can be trivialized over the 2-skeleton. This trivialization extends to the total bundle, since $\pi_2(SO(3)) = 0$ is trivial.

**6.** Let $G \subset U(n)$ be a compact real Lie group with Lie algebra $\mathfrak{g}$.

**a)** Define $\delta : \Lambda^k\mathfrak{g}^* \to \Lambda^{k+1}\mathfrak{g}^*$ by

$$(\delta f)(\xi_1, \ldots, \xi_{k+1}) := \sum_{1\le i<j\le k+1} (-1)^{i+j-1} f([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{k+1}) + \sum_{i=1}^{k} (-1)^i f(\xi_1, \ldots, \hat{\xi}_i, \ldots, x_{k+1}).$$

where $\hat{\xi}_i$ means that the corresponding entry is omitted. Every $f \in \Lambda^k\mathfrak{g}^*$ gives rise to a $k$-form on $\tau_f \in \Omega^k(G)$ defined by

$$(\tau_f)_g(g\xi_1, \ldots, g\xi_k) := f(\xi_1, \ldots, \xi_k).$$

Show that $d\tau_f = \tau_{\delta f}$. Moreover, $\tau_f$ is exact, if and only if there exists $h \in \Lambda^{k-1}\mathfrak{g}^*$ with $f = \delta h$.

**b)** A $k$-form $\omega \in \Omega^k(G)$ is called bi-invariant when

$$L_g^*\omega = \omega = R_g^*\omega \qquad \text{for all } g \in G$$

where $L_g, R_g : G \to G$ are defined by $L_g(h) := gh$ and $R_g(h) := hg$. Denote by $\phi : G \to G$, $\phi(g) := g^{-1}$, the inversion map. Show that

$$\phi^*\omega = (-1)^k\omega \qquad \text{for all bi-invariant } \omega \in \Omega^k(G).$$

Conclude from this that $\omega$ is closed and $[\omega] \in H^k(G)$ represents a non-trivial cohomology class when $\omega \neq 0$.

**7.** Let $G \subset U(n)$ be a compact real Lie group with Lie algebra $\mathfrak{g}$. Define a Riemannian metric on $G$ by

$$\langle g\xi, g\eta \rangle := \frac{1}{2}\mathrm{tr}(\xi^*\eta)$$

for $g \in G$ and $\xi, \eta \in \mathfrak{g} = T_{\mathbb{1}}G$.

**a)** Every abelian Lie group $G$ is a flat Riemannian manifold.

**b)** Show that $\sigma \in \Omega^3(G)$ defined by

$$\sigma_g(g\xi, g\eta, g\zeta) = \langle \xi, [\eta, \zeta] \rangle$$

is a well-defined and bi-invariant 3-form on $G$.

**c)** Suppose $G$ is not abelian, then $H^3(G) \neq 0$.

**d)** Conclude that $S^n$ for $n \neq 0, 1, 3$ cannot be equipped with the structure of a Lie group.

**Hint:** For part a): We have calculated the curvature in the first semester. For part b) use the previous exercise to show that $[\sigma] \in H^3(G)$ is well-defined and non-trivial.

**8.** Let $\pi : E \to M$ and $\pi' : M' \to E'$ be oriented real rank $n$ vector bundles over compact oriented smooth manifolds without boundary. Let $\phi : M' \to M$ and $\Phi : E' \to E$ be smooth maps with $\pi' \circ \Phi = \phi \circ \pi$. Assume that the fibre maps $\Phi|_{E_p} : E_p \to E'_{\phi(p)}$ are orientation preserving vector space isomorphism for every $p \in M$. Prove that

$$\tau(E) = \Phi^*\tau(E').$$

where $\tau(E) \in H^n_c(E)$ and $\tau(E') \in H^n_c(E')$ denote the Thom classes.

**9.** Let $E$ be a rank $m$ vector bundle over a smooth compact $m$-manifold $M$ without boundary. Let $s$ be a section of this bundle with isolated but necessarily transverse zeros. Define the index $i(s,p)$ for every $p \in s^{-1}(0)$ and prove the formula

$$\int_M e(E) = \sum_{s(p)=0} i(s,p)$$

where $e(E) \in \Omega^m(M)$ is the Euler class of $E$.

**10.** The **zeta-function** of a smooth map $f : M \to M$ is given by

$$\zeta_f(t) := \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)t^n}{n}\right), \tag{1}$$

where $L(f^n) = L(f \circ f \circ \cdots f)$ denotes the Lefschetz number.

**a)** Why is the right hand side of the equality well-defined for $t$ sufficiently small, i.e. why does the the sum on the right hand side of (1) converge for small $t$.

**b)** Prove $\det(\mathbb{1} - tA) = \exp\left(-\mathrm{trace}\left(\sum_{n=1}^{\infty} \frac{t^n A^n}{n}\right)\right)$ for all small $t$.

**c)** Using $b$) prove the formula

$$\begin{aligned}\zeta_f(t) &= \prod_{i=1}^{m} \det\Big(\mathbb{1} - tf^* : H^i(M) \to H^i(M)\Big)^{(-1)^{i+1}} \\ &= \frac{\det(\mathbb{1} - tf^* : H^{odd}(M) \to H^{odd}(m))}{\det(\mathbb{1} - tf^* : H^{ev}(M) \to H^{ev}(M))}\end{aligned}$$

**Hint:** For a): For a square matrix $A \in \mathbb{R}^{k\times k}$ it holds $\mathrm{tr}(A)^n = \sum_{i=1}^{k} \lambda_i^n$ where $\lambda_i$ are the generalized eigenvalues of $A$, i.e. the diagonal entries of its Jordan normal form. For b): Recall or proof the formulas $\det(\exp(A)) = \exp(\mathrm{tr}(A))$ and $\log(\mathbb{1} - A) = -\sum_{n=1}^{\infty} \frac{A^n}{n}$ for the matrix exponential and for the matrix logarithm.

**11.** Compute the zeta function of the previous exercise is the following cases.

**a)** Take $f : T^2 \to T^2 : (x, y) \to (ax + by, cx + dy)$ with $a, b, c, d \in \mathbb{Z}$.

**b)** Take $f : S^k \to S^k$ and express the result in terms of $\deg(f)$.

**c)** Take $f = \mathrm{id} : M \to M$ where $M$ is a compact oriented manifold without boundary. Express the result in terms of the Euler characteristic $\chi(M)$.

**12.** Deduce from the Lefschetz fixed point theorem the following.

**a)** The Brouwer fixed point theorem: Every smooth map $f : D^k \to D^k$ has a fixed point.

**b)** Every smooth map $f : \mathbb{R}P^k \to \mathbb{R}P^k$ has a fixed point for $k$ even.

**c)** Prove that for every $f : \mathbb{C}P^k \to \mathbb{C}P^k$, there is a number $d \in \mathbb{Z}$, such that

$$L(f) = 1 + d + d^2 + \dots d^n.$$

Deduce that $f$ has a fixed point when $k$ is even.

**Hint:** For c), use the ring structure of the cohomology of $\mathbb{C}P^k$.

**13.** Let $A \in \mathbb{C}^{n\times n}$ be a square matrix. Prove that $A$ has an eigenvector.

**Hint:** If $\det A \neq 0$, then use $A$ to define a map $\phi_A : \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$. Prove that this map is homotopic to the identity and use this to compute the Lefschetz number of $\phi_A$. Understand what the fixed points of $\phi_A$ are.

**14.** Let $M$ be an compact $m$-manifold without boundary. We call $\omega \in \Omega^2(M)$ a symplectic form if $\omega$ is closed and $\omega$ is non-degenerate. Here non-degenerate means that for $p \in M$ and a non-zero vector $v \in T_pM$, there is a vector $w \in T_pM$ such that $\omega_p(v,w) \neq 0$. Assume $M$ admits a symplectic form.

**a)** Prove that any vector space $V$ with a non-degenerate 2-form $\omega$ admits an isomorphism $\Phi : V \to \mathbb{R}^{2n}$ such that $\Phi^*\omega_0 = \omega$, where

$$\omega_0 := \sum_{i=1}^{n} dx^{2i-1} \wedge dx^{2i}.$$

**b)** Prove that $m$ has to be even. Put $m = 2n$.

**c)** Prove that $\omega$ is non-degenerate if and only if $\omega^n = \omega \wedge \ldots \wedge \omega$ ($n$ times) is a volume form.

**d)** Adapt Moser's argument for volume forms, to prove that given any smooth family of symplectic forms $\omega_t$ with $t \in [0,1]$ and with constant cohomology class $[\omega_t] = a \in H^2(M)$, there is a smooth family of diffeomorphisms $\psi_t$ on $M$ for $t \in [0,1]$ such that

$$\psi_0 = \mathrm{id}, \quad \psi_t^*\omega_1 = \omega_0.$$

You may assume the fact that there is a smooth family $\sigma_t \in \Omega^1(M)$ such that $\frac{d}{dt}\omega_t = d\sigma_t$. This can be proven using Hodge theory, which goes beyond the scope of this course.

**e)** Use Moser's argument to prove that for every point $p \in M$, there is a chart $\varphi_p : U \to \mathbb{R}^{2n}$ such that $\varphi_p^*\omega_0 = \omega$ where $\omega_0$ is as in a). This result is known as Darboux's theorem. It implies that there are no local invariants in symplectic geometry.

Have a nice summer holiday!