Solution 1

1. In this exercise, we will introduce projective spaces. They are an important class of differential manifolds.

a) The real projective plane $\mathbb{R}P^2$ is the set

$$\mathbb{R}P^2 = \{\ell \subset \mathbb{R}^3 : \ell \text{ is a 1-dimensional linear subspace}\}$$

of real lines in $\mathbb{R}^3$. It can be identified with the quotient space

$$\mathbb{R}P^2 = \big(\mathbb{R}^3 \setminus \{0\}\big)/\mathbb{R}^*$$

of nonzero vectors in $\mathbb{R}^3$ modulo the action of the multiplicative group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ of nonzero real numbers. The equivalence class of a nonzero vector $x = (x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{0\}$ will be denoted by

$$[x] = [x_0 : x_1 : x_2] := \{\lambda x : \lambda \in \mathbb{R}^*\}.$$

We define subsets $U_i = \{[x] \in \mathbb{R}P^2 : x_i \neq 0\}$ and charts $\phi_i : U_i \to \mathbb{R}^2$ by $[x] \mapsto (x_j/x_i, x_k/x_i)$ for $i = 0, 1, 2$ and where $\{i, j, k\} = \{0, 1, 2\}$, $j < k$.

(i) Check that $\mathbb{R}P^2$ with the atlas $\{\phi_i\}_{i=0,1,2}$ is a smooth manifold of dimension 2.

(ii) Check that the intrinsic manifold topology is the quotient topology of $\mathbb{R}P^2$

Remark: Note how this definition is more natural than the one seen in exercise 4 of sheet 2 of the first semester.

b) The complex projective space $\mathbb{C}P^1$ is the set

$$\mathbb{C}P^1 = \{\ell \subset \mathbb{C}^2 : \ell \text{ is a 1-dimensional complex linear subspace.}\}$$

of complex lines in $\mathbb{C}^2$. It can be identified with the quotient space

$$\mathbb{C}P^1 = \big(\mathbb{C}^2 \setminus \{0\}\big)/\mathbb{C}^*$$

of nonzero vectors in $\mathbb{C}^2$ modulo the action of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of nonzero complex numbers. The equivalence class of a nonzero vector $z = (z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}$ will be denoted by

$$[z] = [z_0 : z_1] := \{\lambda z : \lambda \in \mathbb{C}^*\}.$$

(i) Find an atlas, which makes $\mathbb{C}P^1$ into a smooth manifold.

(ii) Find an explicit diffeomorphism between $\mathbb{C}P^1$ and the sphere S^2.

Remark: $\mathbb{R}P^n$ and $\mathbb{C}P^n$ can be defined similarly for all $n \geq 0$.

Solution:

a) (i) We see that the maps ϕ_i are bijective for $i = 0, 1, 2$. The inverse for $i = 0$ is given for example by $\phi_0^{-1}(x_1, x_2) = [1 : x_1 : x_2]$. Furthermore, $\phi_i(U_i \cap U_j) = \{x \in \mathbb{R}^2 : x_j \neq 0\}$ for $i \neq j$ is open. The transition maps are given by

$$\phi_{10} : (\mathbb{R}\setminus\{0\}) \times \mathbb{R} \to (\mathbb{R}\setminus\{0\}) \times \mathbb{R}, \qquad \phi_{10}(x_1, x_2) = \left(\frac{1}{x_1}, \frac{x_2}{x_1}\right)$$

$$\phi_{20} : (\mathbb{R}\setminus\{0\}) \times \mathbb{R} \to \mathbb{R} \times (\mathbb{R}\setminus\{0\}), \qquad \phi_{20}(x_1, x_2) = \left(\frac{1}{x_2}, \frac{x_1}{x_2}\right)$$

$$\phi_{21} : \mathbb{R} \times (\mathbb{R}\setminus\{0\}) \to \mathbb{R} \times (\mathbb{R}\setminus\{0\}), \qquad \phi_{21}(x_1, x_2) = \left(\frac{x_1}{x_2}, \frac{1}{x_2}\right)$$

and their inverses, which are all smooth. Thus $\mathcal{A} = \{(\phi_0, U_0), (\phi_1, U_1), (\phi_2, U_2)\}$ is a smooth atlas. Thus $\mathbb{R}P^2$ is thus given the structure of a smooth manifold structure together with an intrinsic manifold topology.

(ii) We now check that this topology is equal to the quotient topology.

Denote by $\pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}P^2 : x \mapsto [x]$ the projection. We start by noting that $\pi^{-1}(U_i) \subset \mathbb{R}^3 \backslash \{0\}$ are open for $i = 0, 1, 2$. In particular, $U_i \subset (\mathbb{R}^3 \backslash \{0\})/\mathbb{R}^*$ are open with respect to the quotient topology and $W \subset (\mathbb{R}^3 \backslash \{0\})/\mathbb{R}^*$ is open if and only if $W_i := W \cap U_i$ are open for $i = 0, 1, 2$.

Now $W_0 = W \cap W_0 \subset U_0 \subset \mathbb{R}P^2$ is open in the intrinsic manifold topology if and only if

$$\phi_0(W_0) = \{(x_1, x_2) \,:\, [1 : x_1 : x_2]\} \subset \mathbb{R}^2$$

is open in $\mathbb{R}^2$. And W_0 is open in the quotient topology if and only if

$$\pi^{-1}(W_0) = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \backslash \{0\} \,|\, [x_0 : x_1 : x_2] \in W_0\}$$

is open. The equivalence of these two conditions is somewhat obvious and can be verified directly by looking at ϵ-neighborhoods. Alternatively, one could argue using the map

$$f_0 : \mathbb{R}^* \times \mathbb{R}^2 \to \pi^{-1}(U_0), \qquad (\lambda, (x_1, x_2)) \mapsto (\lambda, \lambda x_1, \lambda x_2).$$

and verify that it is a homeomorphism which satisfies $f_0(\mathbb{R}^* \times \phi_0(W_0)) = \pi^{-1}(W_0)$.

The case $i = 1, 2$ works similarly.

b) (i) We take two charts

$$U_0 = \{[z] \in \mathbb{C}P^1 : z_0 \neq 0\}, \qquad \phi_0 : U_0 \to \mathbb{R}^2 \cong \mathbb{C} : [z_0, z_1] \mapsto z_1/z_0,$$
$$U_1 = \{[z] \in \mathbb{C}P^1 : z_1 \neq 0\}, \qquad \phi_1 : U_1 \to \mathbb{R}^2 \cong \mathbb{C} : [z_0, z_1] \mapsto z_0/z_1,$$

where z/w stands for division of complex numbers. The inverses are given by

$$\phi_0^{-1} : \mathbb{R}^2 \to U_0, \quad z \mapsto [1 : z], \quad \phi_1^{-1} : \mathbb{R}^2 \to U_1, \quad z \mapsto [z : 1].$$

Thus the transition maps are given by

$$\phi_{10} : \phi_0(U_0 \cap U_1) = \mathbb{C} \setminus \{0\} \to \phi_1(U_0 \cap U_1) = \mathbb{C} \setminus \{0\}, \qquad z \mapsto 1/z$$

and so is smooth. Thus $\mathbb{C}P^1$ together with the atlas $\mathcal{A} = \{(\phi_0, U_0), (\phi_1, U_1)\}$ is a smooth manifold.

(ii) Let $p_N := (0, 0, 1)$ and $p_S := (0, 0, -1)$ denote the north and south pole of S^2. Stereographic projection from p_N and p_S defines charts on S^2 given by the formula

$$\phi_N : S^2 \backslash \{p_N\} \to \mathbb{R}^2 : (x_0, x_1, x_2) \mapsto \left(\frac{x_0}{1 - x_2}, \frac{x_1}{1 - x_2}\right)$$
$$\phi_S : S^2 \backslash \{p_S\} \to \mathbb{R}^2 : (x_0, x_1, x_2) \mapsto \left(\frac{x_0}{1 + x_2}, \frac{x_1}{1 + x_2}\right)$$

and $\mathcal{A} := \{\phi_N, \Phi_S\}$ is an atlas for S^2.

We define the diffeomorphism by

$$\Phi : \mathbb{C}P^1 \to S^2, \qquad \phi([z_0, z_1]) := \begin{cases} \phi_N^{-1}(z_0/z_1) & z_1 \neq 0 \\ p_N & z_1 = 0 \end{cases}$$

Here $z_1/z_2 \in \mathbb{C} \cong \mathbb{R}^2$ is understood as vector in $\mathbb{R}^2$, so that we can plug it into ϕ_N^{-1}. This map is clearly bijective. To show that it is a smooth diffeomorphism, we need to look at it in charts:

$$\phi_N \circ \Phi \circ \phi_0^{-1} : \mathbb{C} \to \mathbb{R}^2, \qquad (x + \mathbf{i}y) \mapsto (x, y)$$

is obviously smooth with smooth inverse. (That is how we defined Φ). The non-trivial verification is that $\phi_S \circ \Phi \circ \phi_1^{-1}$ is smooth. In complex coordinates we have

$$\phi_N^{-1} : \mathbb{C} \cong \mathbb{R}^2 \to S^2 \backslash \{p_N\} \subset \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}, \qquad \phi_N^{-1}(z) = \left(\frac{2z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$$

and hence for $z \neq 0$:

$$\begin{aligned}\phi_S \circ \Phi \circ \phi_1^{-1}(z) = \phi_S \circ \phi_N^{-1}\left(\frac{1}{z}\right) &= \phi_S\left(\frac{2/z}{|z|^{-2}+1}, \frac{|z|^{-2}-1}{|z|^{-2}+1}\right) \\ &= \phi_S\left(\frac{2\bar{z}}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right) = \bar{z}.\end{aligned}$$

For $z = 0$ we have $(\phi_S \circ \Phi \circ \phi_1^{-1})(0) = \phi_S(p_N) = 0$ and the formula remains valid. I.e. we have shown:

$$\phi_S \circ \Phi \circ \phi_1^{-1} : \mathbb{C} \to \mathbb{C}, \qquad z \mapsto \bar{z}$$

and this is a smooth map with smooth inverse.

2. **a)** We define $\varphi_1 : \mathbb{R} \to \mathbb{R} : x \mapsto x$ and $\varphi_2 : \mathbb{R} \to \mathbb{R} : x \mapsto x^3$. Prove that $\mathcal{A}_i = \{\varphi_i\}$ are smooth atlases for $\mathbb{R}$ with $i = 1, 2$. Prove that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is not an atlas. Prove that $(\mathbb{R}, \mathcal{A}_1)$ is diffeomorphic to $(\mathbb{R}, \mathcal{A}_2)$.

Remark: It can be proven that in dimension 1,2 and 3, every underlying topological space of a smooth manifold has only one smooth manifold structure up to diffeomorphism. It is an open question whether S^4 has a unique differentiable structure. A remarkable result by Donaldson give uncountably many non-diffeomorphic smooth structures on $\mathbb{R}^4$. Another result by Milnor says that S^7 allows exactly 28 different smooth structures up to diffeomorphisms.

b) Prove that any submanifold M of $\mathbb{R}^n$ as seen in the last semester is also an intrinsic manifold.

Remark: The Whitney Embedding Theorem gives a converse to this statement. Namely, for every intrinsic m-manifold M, there is an embedding as submanifold into $\mathbb{R}^n$ for $n = 2m$. So intrinsic manifolds are the same concept as submanifolds in $\mathbb{R}^n$ from last semester. However this semester, in the context of differential topology, we only want to study properties of manifolds up to diffeomorphisms and not manifolds and their specific embedding into $\mathbb{R}^n$. This is why we need the more abstract concept of intrinsic manifolds.

Solution:

a) Both atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ have only one bijective map, so these are smooth atlases. However, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is not an atlas since the transition map $\phi_2 \circ \phi_1^{-1}(x) = x^3$ which is not a diffeomorphism.

This simply means that the identity map $(\mathbb{R}, \mathcal{A}_2) \to (\mathbb{R}, \mathcal{A}_1), x \mapsto x$, is not a diffeomorphism for these two smooth structures. Nevertheless, both smooth structures are diffeomorphic and a diffeomorphism $\Phi : (\mathbb{R}, \mathcal{A}_2) \to (\mathbb{R}, \mathcal{A}_1)$ is given by $\Phi(x) = x^3$. (This holds by definition since $(\phi_1 \circ \Phi \circ \phi_2^{-1})(x) = x$ is a diffeomorphism of $\mathbb{R}$.)

b) Recall that a subset $M \subset \mathbb{R}^n$ is a smooth m-dimensional submanifold when for every $p \in M$ there exists an open neighborhood $U_p \subset M$ (in the relative topology) and a diffeomorphism

$$\phi_p : U_p \to \Omega_p \subset \mathbb{R}^m$$

onto an open subset of $\mathbb{R}^m$. Recall that we generalized the notion of smooth maps and diffeomorphism for this to make sense: A function defined on an arbitrary subset of $\mathbb{R}^n$ is smooth when it is the restriction of a smooth function defined on an open neighborhood and a diffeomorphism is a bijective map which is smooth and has a smooth inverse. Now given two charts the ϕ_p and ϕ_q with $U_p \cap U_q \neq \emptyset$ the transition map

$$\phi_p \circ \phi_q^{-1} : \phi_q(U_p \cap U_q) \to \phi_p(U_p \cap U_q)$$

is clearly bijective and smooth (as composition of smooth and bijective functions). Hence the collection of all these charts defines an atlas $\mathcal{A} = \{(\phi_p, U_p)\}_{p \in M}$ for M. It is not hard to check that its intrinsic topology is the subset topology.

3. **a)** We identify $\mathbb{C}P^1$ with $\mathbb{C} \cup \{\infty\}$ by $[z : 1] \mapsto z$ and $[1 : 0] \mapsto \infty$. Consider the map $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ given by $z \mapsto \frac{1}{z}$ for $z \neq 0$ and $0 \mapsto \infty$. Prove that this defines a smooth diffeomorphism $f : \mathbb{C}P^1 \to \mathbb{C}P^1$.

b) Prove that the map $g : S^2 \to \mathbb{R}P^2 : (x_0, x_1, x_2) \mapsto [x_0 : x_1 : x_2]$ is a smooth double cover.

Solution:

a) Written as a map from $\mathbb{C}P^1 \to \mathbb{C}P^1$, we have $f([z_0 : z_1]) = [z_1 : z_0]$. We see that $f(U_0) = U_1$, $f(U_1) = U_0$ and

$$\varphi_0 \circ f \circ \varphi_1^{-1}(z) = z, \qquad \text{and} \qquad \varphi_1 \circ f \circ \varphi_0^{-1}(z) = z.$$

Since the identity is smooth and $U_0 \cup U_1 = \mathbb{C}P^1$, f is smooth. Since $f(f(z)) = z$, we have $f^{-1} = f$ is also smooth and thus we get a diffeomorphism.

b) For S^2, we choose the atlas with six charts defined on

$$V_i^{\pm} = \{p = (x_0, x_1, x_2) \in S^2 : \pm x_i > 0\}$$

for $i = 0, 1, 2$ and given by

$$\begin{aligned} \varphi_0^{\pm} : V_0^{\pm} \to B_1(0) \subset \mathbb{R}^2 : &\qquad \varphi^{\pm}(x_0, x_1, x_2) := (x_1, x_2), \\ \varphi_1^{\pm} : V_1^{\pm} \to B_1(0) \subset \mathbb{R}^2 : &\qquad \varphi^{\pm}(x_0, x_1, x_2) := (x_0, x_2), \\ \varphi_2^{\pm} : V_2^{\pm} \to B_1(0) \subset \mathbb{R}^2 : &\qquad \varphi^{\pm}(x_0, x_1, x_2) := (x_0, x_1). \end{aligned}$$

Their inverses are given by

$$\begin{aligned} (\varphi_0^{\pm})^{-1} : B_1(0) \to S^2 : (x_1, x_2) &\mapsto \left(\pm\sqrt{1 - x_1^2 - x_2^2}, x_1, x_2\right), \\ (\varphi_1^{\pm})^{-1} : B_1(0) \to S^2 : (x_1, x_2) &\mapsto \left(x_1, \pm\sqrt{1 - x_1^2 - x_2^2}, x_2\right), \\ (\varphi_2^{\pm})^{-1} : B_1(0) \to S^2 : (x_1, x_2) &\mapsto \left(x_1, x_2 \pm \sqrt{1 - x_1^2 - x_2^2}\right), \end{aligned}$$

For $\mathbb{R}P^2$ we take the atlas $\mathcal{A} = \{(\phi_i, U_i)\}_{i=0,1,2}$ which was defined in Exercise 1.

Now assume that $p \in V_0^+$, then $g(V_0^+) \subset U_0$. Thus we can look at the composition

$$\phi_0 \circ g \circ \left(\varphi_0^+\right)^{-1}(x_1, x_2) = \left(\frac{x_1}{\sqrt{1 - x_1^2 - x_2^2}}, \frac{x_2}{\sqrt{1 - x_1^2 - x_2^2}}\right)$$

which defines a smooth map. The same goes for $\phi_0 \circ g \circ (\varphi_0^-)^{-1}$, $\phi_1 \circ g \circ (\varphi_1^{\pm})^{-1}$ and $\phi_2 \circ g \circ (\varphi_2^{\pm})^{-1}$. So g is smooth and furthermore it is a double cover.

4. Let M be a manifold with an atlas $\mathcal{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$. We give the following two definitions of the tangent space at $p \in M$.

(i) Two smooth curves $\gamma_0, \gamma_1 : \mathbb{R} \to M$ with $\gamma_0(0) = \gamma_1(0) = p$ are called p-equivalent if for some (and hence every) $\alpha \in A$ with $p \in U_\alpha$ we have

$$\frac{d}{dt}\bigg|_{t=0} \phi_\alpha(\gamma_0(t)) = \frac{d}{dt}\bigg|_{t=0} \phi_\alpha(\gamma_1(t)).$$

We write $\gamma_0 \overset{p}{\sim} \gamma_1$ if γ is p-equivalent to γ_1 and denote the equivalence class of a smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$ by $[\gamma]_p$. The tangent space of M at p is the set of equivalence classes

$$T_pM := \{[\gamma]_p : \gamma : \mathbb{R} \to M \text{ is smooth and } \gamma(0) = p\}.$$

(ii) The $\mathcal{A}$-tangent space of M at p is the quotient space

$$T_p^{\mathcal{A}}M := \bigcup_{p \in U_\alpha} \{\alpha\} \times \mathbb{R}^m / \sim_p,$$

where the union runs over all $\alpha \in A$ with $p \in U_\alpha$ and

$$(\alpha, \xi) \sim_p (\beta, \eta) \iff d(\phi_\beta \circ \phi_\alpha^{-1})(x)\xi = \eta, x := \phi_\alpha(p).$$

The equivalence class will be denoted by $[\alpha, \xi]_p$.

Show that the natural map

$$T_pM \mapsto T_p^{\mathcal{A}}M : [\gamma]_p \mapsto \left[\alpha, \frac{d}{dt}\bigg|_{t=0} \phi_\alpha(\gamma(t))\right]$$

is well-defined and bijective.

Since $T_p^{\mathcal{A}}M$ has a canonical vector space structure of dimension m, this bijection induces a vector space structure on the set T_pM. This exercise shows that both definitions of tangent space are equivalent.

Solution: We check that the map is well-defined: By definition of p-equivalence, the image

$$\left[\alpha, \frac{d}{dt}\bigg|_{t=0} \phi_\alpha(\gamma(t))\right] \in T_p^{\mathcal{A}}M$$

does not depend on the representative γ of the equivalence class $[\gamma]_p$. We show that it is also independent of the choice of the chart domain U_α with $p \in U_\alpha$. Thus assume that $p \in U_\beta$ for some other chart domain. Then it gets assigned to

$$\left[\beta, \frac{d}{dt}\bigg|_{t=0} \phi_\beta(\gamma(t))\right] \in T_p^{\mathcal{A}}M$$

and by the chain-rule it holds

$$\frac{d}{dt}\bigg|_{t=0} \phi_\beta(\gamma(t)) = d(\phi_\beta \circ \phi_\alpha^{-1})(\phi(p)) \frac{d}{dt}\bigg|_{t=0} \phi_\alpha(\gamma(t)).$$

So both images are represent the same equivalence class

$$\left[\alpha, \frac{d}{dt}\bigg|_{t=0} \phi_\alpha(\gamma(t))\right] = \left[\beta, \frac{d}{dt}\bigg|_{t=0} \phi_\beta(\gamma(t))\right] \in T_p^{\mathcal{A}}M$$

and hence the isomorphism does not depend on the choice of the chart.

We show injectivity: Let γ_0 and γ_1 be two curves with $\gamma_0(0) = p = \gamma_1(p)$ and assume that $[\gamma_0]_p$ and $[\gamma_1]_p$ gets assigned to the same element in $T_p^{\mathcal{A}}M$. This is equivalent to

$$\frac{d}{dt}\bigg|_{t=0} \phi_\alpha(\gamma_0(t)) = \frac{d}{dt}\bigg|_{t=0} \phi_\alpha(\gamma_1(t))$$

and hence $\gamma_0 \sim_p \gamma_1$ are p-equivalent. In particular $[\gamma_0]_p = [\gamma_1]_p$ and the map is injective. We show surjectivity: Let $[\alpha, v] \in T_p^{\mathcal{A}} M$ be given: For $\epsilon > 0$ sufficiently small the curve

$$\tilde{\gamma} : (-\epsilon, \epsilon) \to M, \qquad \tilde{\gamma}(t) := \phi_\alpha^{-1}(tv)$$

is well-defined. Now choose smooth function $\rho : \mathbb{R} \to (-\epsilon, \epsilon)$ with $\rho'(0) = 1$ (e.g. $\rho(t) = \epsilon \sin(t/\epsilon)$ or a combination of suitable cuttoff functions) and define

$$\gamma : \mathbb{R} \to M, \qquad \gamma(t) := \tilde{\gamma}(\rho(t)).$$

Then $[\gamma]_p$ gets assigned to $[\alpha, v]$ and this proves surjectivity.

5. Let M be a m-manifold with an atlas $\mathcal{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$.

a) Define the tangent bundle as

$$TM = \bigcup_{p \in M} \{p\} \times T_p M,$$

and denote by $\pi : TM \to M$ the projection given by $\pi(p, v) := p$.

Prove that TM is a smooth $2m$-dimensional manifold with atlas consisting of the charts

$$\tilde{\phi}_\alpha : \tilde{U}_\alpha := \pi^{-1}(U_\alpha) \to \phi_\alpha(U_\alpha) \times \mathbb{R}^m : (p, v) \mapsto (\phi_\alpha(p), d\phi_\alpha(p)v)$$

for $\alpha \in A$. Also, verify that $\pi : TM \to M$ is a smooth submersion.

b) We give the following two definitions of vector fields.

(i) A map $X : M \to TM$ is a vector field if X is smooth and $\pi \circ X = \mathrm{id}$. Denote the set of such maps X by $\mathrm{Vect}(M)$.

(ii) A collection of smooth maps $X_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^m$ for $\alpha \in A$ is called $\mathcal{A}$-vector field if for all $\alpha, \beta \in A$ and all $x \in \phi_\alpha(U_\alpha \cap U_\beta)$, we have $X_\beta(\phi_{\beta\alpha}(x)) = d\phi_{\beta\alpha}(x) X_\alpha(x)$. Denote the set of such collections $\{X_\alpha\}_{\alpha \in A}$ by $\mathrm{Vect}^{\mathcal{A}}(M)$.

Define the map $\mathrm{Vect}(M) \to \mathrm{Vect}^{\mathcal{A}}(M)$ by

$$X \mapsto \{X_\alpha\}_{\alpha \in A} \text{ where } X_\alpha(x) := d\phi_\alpha(p) X(p),\ p := \phi_\alpha^{-1}(x).$$

Prove that this map is well-defined and bijective.

Solution:

a) For TM to be a manifold, we only need to verify that transition maps are smooth: For $\alpha, \beta \in A$, we have

$$\begin{aligned} \tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1} : \phi_\beta\left(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)\right) &\to \phi_\alpha\left(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)\right) \\ (x, \xi) &\mapsto \left((\phi_\alpha \circ \phi_\beta^{-1})(x),\, d(\phi_\alpha \circ \phi_\beta^{-1})(x)\xi\right). \end{aligned}$$

Since $\phi_\alpha \circ \phi_\beta^{-1}$ are the transition maps for the atlas on M, they are by assumption smooth, and hence the transition maps $\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1}$ are also smooth.

We remark that this smooth manifold structure on TM agrees for a submanifold $M \subset \mathbb{R}^n$ with that of the submanifold $\{(p, v) \in \mathbb{R}^n \times \mathbb{R}^n \ :\ p \in M,\, v \in T_p M\}$. Indeed, we saw in the last semester that

$$\phi_\alpha(U_\alpha) \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \qquad (x, \xi) \mapsto (\phi_\alpha(x), d\phi_\alpha(x)\xi)$$

are charts for the embedded tangent bundle.

We verify that $\pi : TM \to M$ is submersion. Indeed, let $p \in M$ and choose a chart domain with $p \in U_\alpha$. On the open set $\pi^{-1}(U_\alpha)$ we can express π in local coordinates using the charts $\tilde{\phi}_\alpha$ and ϕ_α as

$$\phi_\alpha \circ \pi \circ \tilde{\phi}_\alpha : \phi_\alpha(U_\alpha) \times \mathbb{R}^m \to \phi_\alpha(U_\alpha), \qquad (x, \xi) \mapsto x.$$

This map is clearly smooth and its differential is surjective at every point (x, ξ). This proves that π is a submersion.

b) Let $X : M \to T^{\mathcal{A}}M$ be a smooth map with $\pi(X(p)) = p$. In local coordinates we can express this map as

$$\tilde{\phi}_\alpha \circ X \circ \phi_\alpha^{-1} : \qquad x \mapsto (x, X_\alpha(x)).$$

where $X_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^m$ is a smooth function. Here smoothness follows from the assumption that X itself is smooth. Equivalently, we have that

$$X(p) = [\alpha, X_\alpha(\phi_\alpha(p))] \in T_p^{\mathcal{A}}M$$

for all $p \in U_\alpha$. Since $X(p) \in T_pM$ does not depend on the choice of particular charts, we must have

$$[\alpha, X_\alpha(\phi_\alpha(p))] = X(p) = [\beta, X_\beta(\phi_\beta(p))]$$

and so the local functions $\{X_\alpha\}_{\alpha \in A}$ satisfy the compatibility condition $X_\beta((\phi_\beta \circ \phi_\alpha^{-1})(x)) = d(\phi_\beta \circ \phi_\alpha^{-1})(x)X_\alpha(x)$ for any $x \in \phi_\alpha(U_\alpha \cap U_\beta)$.

Conversely, given such a collection of local functions $\{X_\alpha\}_{\alpha \in A}$, we can simply define $X(p) := [\alpha, X_\alpha(\phi_\alpha(p))]$ for any chart with $p \in U_\alpha$. The compatibility condition guarantees that X is well-defined and smoothness of all the X_α translates into smoothness of X.

6. **a)** Prove that there exists a canonical isomorphism $T_\ell\mathbb{R}P^n \cong \mathcal{L}(\ell, \ell^\perp)$ where $\perp$ is the orthogonal complement with respect to the Euclidean metric on $\mathbb{R}^{n+1}$.

b) Prove that the tangent bundle $T\mathbb{T}^3$ of the three torus is diffeomorphic to $\mathbb{T}^3 \times \mathbb{R}^3$. Characterise vector fields on $\mathbb{T}^3$.

Solution:

a) We present the geometric idea of the isomorphism: Given $\hat{\ell} \in T_\ell\mathbb{R}P^n$ we can represent it by a smooth curve $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}P^n$ with $\gamma(0) = \ell$ and $\dot{\gamma}(0) = \hat{\ell}$. (In the notation of Exercise 4: $\hat{\ell} = [\gamma]_\ell$.) For every $x \in \ell$ there exists a unique smooth lift $\tilde{\gamma}_x$ satisfying

$$\pi(\tilde{\gamma}(t)) = \gamma(t), \qquad \gamma(0) = x, \qquad ||\gamma(t)|| = ||x||$$

where $\pi : \mathbb{R}^{n+1}\backslash\{0\} \to (\mathbb{R}^{n+1}\backslash\{0\})/\mathbb{R}^* = \mathbb{R}P^n$ is the canoncial projection. Then

$$\Phi_{\hat{\ell}} : \ell \to \ell^\perp, \qquad \Phi_{\hat{\ell}}(x) := \dot{\tilde{\gamma}}_x(0)$$

is a well-defined linear map and

$$T_\ell\mathbb{R}P^n \to \mathcal{L}(\ell, \ell^\perp), \qquad \hat{\ell} \mapsto \Phi_{\hat{\ell}}$$

is a linear isomorphism. Of course here is much to check: First the existence and uniqueness of the smooth lifts $\tilde{\gamma}_x$, second $\Phi_{\hat{\ell}}$ does not depend on the particular curve γ which represents $\hat{\ell}$, third $\Phi_{\hat{\ell}}$ is well-defined, and finally $\hat{\ell} \mapsto \Phi_{\hat{\ell}}$ is a linear isomorphism. All of this is straightforward (and maybe a good exercise), but quite tedious and we do not give all the details here.

For the inverse map let $\Phi \in \mathcal{L}(\ell, \ell^\perp)$ be given and define

$$\gamma_\Phi : \mathbb{R} \to \mathbb{R}P^n, \qquad \gamma(t) := \{x + t\Phi(x) \,|\, x \in \ell\}.$$

Then γ is a smooth curve with $\gamma(0) = 0$ and we shall show that $\Phi \mapsto \dot{\gamma}_\Phi(0)$ defines an isomorphism between $\mathcal{L}(\ell, \ell^\perp)$ and $T_\ell \mathbb{R}P^n$. For this recall that an atlas for $\mathbb{R}P^n$ is given by

$$\phi_i : U_i := \{[x_0, \ldots, x_n] \in \mathbb{R}P^n \;:\; x_i \neq 0\} \to \mathbb{R}^n$$
$$\phi_i([x_0, \ldots, x_n]) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$$

for $i = 0, 1, \ldots, n$. Assume w.l.o.g that $\ell = [1 : x_1 : \cdots : x_n] \in U_0$ and define

$$\xi := (\xi_0, \ldots, \xi_n) := \Phi(1, x_1, \ldots, x_n) \in \ell^\perp \subset \mathbb{R}^{n+1}$$

Then follows

$$\gamma_\Phi(t) = [1 + t\xi_0 : x_1 + t\xi_1 : \ldots : x_n + t\xi_n]$$

and

$$(\phi_0 \circ \gamma_\Phi)(t) = \left(\frac{x_1 + t\xi_1}{1 + t\xi_0}, \ldots, \frac{x_1 + t\xi_1}{1 + t\xi_0}\right).$$

In particular, with respect to the chart ϕ_0 the tangent vector $\dot{\gamma}_\Phi(0)$ is represented by

$$\left.\frac{d}{dt}\right|_{t=0} (\phi_0 \circ \gamma_\Phi)(t) = (\xi_1 - x_1\xi_0, \xi_2 - x_2\xi_0, \ldots, \xi_n - x_n\xi_0).$$

This is the zero vector if and only if $\xi_j = x_j\xi_0$ for $j = 1, \ldots, n$. The later is equivalent to $\xi = \xi_0(1, x_1, \ldots, x_n) \in \ell$ and since $\xi \in \ell^\perp$ by assumption this yields $\xi = 0$. Hence the map $\mathcal{L}(\ell, \ell^\perp) \to T_\ell\mathbb{R}P^n$ is injective. Since $\dim(\mathcal{L}(\ell, \ell^\perp)) = n$, its is then also bijective and this proves the claim.

b) We consider the standard embedded flat torus $\mathbb{T}^3 = (S^1)^3 \subset \mathbb{C}^3$. Its tangent bundle is naturally embedded in $\mathbb{C}^3 \times \mathbb{C}^3$ as

$$T\mathbb{T}^3 = \left\{(z_1, z_2, z_3, \zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^6 \,|\, |z_1| = |z_2| = |z_3| = 1,\, \zeta_1/z_1, \zeta_2/z_2, \zeta_3/z_3 \in \mathbf{i}\mathbb{R}\right\}.$$

This is diffeomorphic to $\mathbb{T}^3 \times \mathbb{R}^3$ where an explicit diffeomorphism is given by

$$\mathbb{T}^3 \times \mathbb{R}^3 \to T\mathbb{T}^3, \qquad (z_1, z_2, z_3, t_1, t_2, t_3) \mapsto (z_1, z_2, z_3, \mathbf{i}z_1t_1, \mathbf{i}z_2t_2, \mathbf{i}z_3t_3).$$

We present an alternative intrinsic answer: The flat torus can be described as quotient $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$. This quotient space is a manifold with the following atlas: Denote by $\pi : \mathbb{R}^3 \to \mathbb{R}^3/\mathbb{Z}^3$ the natural projection and choose a collection $\{U_\alpha\}_{\alpha\in A}$ of open subsets of $\mathbb{R}^3$ with $\sup_{x,y\in U_\alpha} |x - y| < 1$ and such that $\{\pi(U_\alpha)\}_{\alpha\in A}$ covers $\mathbb{T}^3$. Then the restrictions $\pi|_{U_\alpha}$ are all injective and thus

$$\phi_\alpha : \pi(U_\alpha) \to U_\alpha, \qquad \phi_\alpha(\pi(x)) := \left(\pi|_{U_\alpha}\right)^{-1}(\pi(x))$$

are well-defined and bijective. The transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ are constant translations (i.e. they are of the shape $x \mapsto x+k$ for some $k \in \mathbb{Z}^3$). In particular they are smooth and the collection $\mathcal{A} = \{\phi_\alpha\}_{\alpha \in A}$ of all these charts defines an atlas for $\mathbb{T}^3$. Note that the tangent space of $\pi(x)$ can be defined as

$$T_{\pi(x)}\mathbb{T}^3 = \bigcup_{\pi(x) \in U_\alpha} \alpha \times \mathbb{R}^3 / \sim$$

where the equivalence relation is given by the derivative of the transition maps. Since the transition maps are constant translations, their derivative is the identity and we get $(\alpha, \xi) \sim (\beta, \xi)$. Hence we have a canonical map $\mathbb{R}^3 \to T_{\pi(x)}\mathbb{T}^3$ for every $\pi(x) \in \mathbb{T}^3$ and this yields the desired isomorphism:

$$\Phi : \mathbb{T}^3 \times \mathbb{R}^3 \to T\mathbb{T}^3, \qquad \pi(x) \mapsto (\pi(x), [\alpha_{\pi(x)}, \xi])$$

where $\alpha_{\pi(x)}$ is any choice of chart with $\pi(x) \in U_\alpha$. This map is clearly bijective and when expressed in charts of $\mathbb{T}^3 \times \mathbb{R}^3$ and $T\mathbb{T}^3$ it is either the identity or a translation in the first three coordinates. In particular it is smooth and Φ is indeed a diffeomorphism.