Solution 2

1. If $m < p$, show that every smooth map $f : M^m \to S^p$ is homotopic to a constant map.

Solution: As $m < p$, every point of M is critical in the sense that $df(x) : T_xM \to T_{f(x)}S^p$ is not surjective. It follows from Sard's theorem that $f(M)$ (the set of critical values) has measure zero. This implies that there exists a point $q \in S^p$ such that $q \notin f(M)$. Use the stereographic projection $\pi : S^p \setminus \{q\} \to \mathbb{R}^p$ to define

$$H : M \times [0,1] \to S^q, \qquad H(x,t) = \pi^{-1}(t(\pi \circ f)(x))$$

This is a smooth homotopy satisfying $H(x,1) = f(x)$ and $f(x,0) = \pi^{-1}(0) = -q$ is a constant map.

2. If two maps f and g from M to S^k satisfy $\|f(x) - g(x)\| < 2$ for all x, prove that f is homotopic to g, the homotopy being smooth if f and g are smooth.

Solution: We can construct a homotopy between f and g using a unique shortest geodesic connecting $f(x)$ and $g(x)$. On the sphere $S^k \subset \mathbb{R}^{k+1}$, we can describe it even more precisely.

a) Take the line segment ℓ connecting $f(x)$ and $g(x)$. Notice that $0 \notin \ell$, because $\|f(x) - g(x)\| < 2$ and so $f(x)$ and $g(x)$ are never antipodal points.

b) Project radially to the sphere by the map $\mathbb{R}^{k+1} \setminus \{0\} \to S^k$, $x \mapsto \frac{x}{\|x\|}$.

The desired homotopy $H : M \times [0,1] \to S^k$ is given by

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}.$$

This is clearly smooth and satisfies $H(x,0) = f(x)$ and $H(x,1) = g(x)$. It is well-defined by our geometric argument in (a). We shall nevertheless backup this up by a direct calculation: The triangle inequality yields

$$\|(1-t)f(x) + tg(x)\| = \|f(x) + t(g(x) - f(x))\| \geq 1 - t\|g(x) - f(x)|$$

$$\|(1-t)f(x) + tg(x)\| = \|g(x) + (1-t)(f(x) - g(x))\| \geq 1 - (1-t)\|g(x) - f(x)|$$

Using the first equation for $0 \leq t \leq \frac{1}{2}$ and the second equation for $\frac{1}{2} \leq t \leq 1$, it follows

$$\|(1-t)f(x) + tg(x)\| \geq 1 - \frac{1}{2}\|f(x) - g(x)\|$$

and hence H is well-defined provided that $\|f(x) - g(x)\| < 2$.

3. Prove that any compact manifold M^k can be embedded into R^{2k+1}.

Hint: Use or reprove the fact that M can be embedded into $\mathbb{R}^N$ for some large k, $f : M \hookrightarrow \mathbb{R}^N$. Prove that the projection parallel to v, $\|v\| = 1$, $\pi_v : \mathbb{R}^N \to v^\perp = H$, $\pi_v(x) = x - \langle x, v\rangle v$ induces an embedding of $f(M)$, for some v. To prove this use Sard's theorem with the following auxiliary maps:

$$F : M \times M \times \mathbb{R} \to \mathbb{R}^N, \ \ F(p,q,t) = t(f(p) - f(q)), \quad G : TM \times \mathbb{R} \to \mathbb{R}^N, \ \ G(p,v,t) = tdf(p)v.$$

Solution: We first sketch the proof of the fact that any compact manifold M can be embedded into some $\mathbb{R}^N$.

For this choose a finite atlas $\{(U_i, \varphi_i)\}_{i=1,\ldots,K}$ of M and a partition of unity subordinate to this atlas: That is a collection of smooth functions $\{\rho_i : M \to [0,1]\}_{i=1,\ldots,K}$ satisfying

(i) $\operatorname{supp}(\rho_i) \subset U_i$ for $i = 1, \ldots, K$,

(ii) $\sum_{i=1}^K \rho_i(x) = 1$ for all $x \in M$.

Next define the functions

$$\Psi_i : M \to \mathbb{R}^k, \qquad \Psi_i(x) := \begin{cases} \rho_i(x)\varphi_i(x), & \text{for } x \in U_i \\ 0 & \text{for } x \notin U_i \end{cases}$$

for $i = 1, \ldots, K$. Finally, we define our embedding by

$$\mathcal{F} : M \to \mathbb{R}^{K(k+1)} : x \mapsto (\Psi_1(x), \ldots, \Psi_i(x), \ldots, \Psi_K(x), \rho_1(x), \ldots, \rho_i(x), \ldots, \rho_K(x)).$$

A moment's thought shows that $\mathcal{F}$ is an injective immersion. Since M is compact, it automatically proper and thus $\mathcal{F}$ is an embedding.

Next, the idea is that is we have an embedding $f : M \to \mathbb{R}^N$ for $N > 2k+1$, then the embedded manifold can be projected onto a hyperplane of lower dimension. Repeating this step sufficiently often yields an embedding $\tilde{f}_v : M \to \mathbb{R}^{2k+1}$ at which point the trick ceases to work.

For $v \in S^{N-1}$, we define the projection

$$\pi_v : \mathbb{R}^k \to v^\perp,\ x \mapsto x - \langle v, x\rangle v.$$

We will prove that there is some $v \in S^{N-1}$ such that $\tilde{f}_v = \pi_v \circ f : M \to v^\perp \cong \mathbb{R}^{N-1}$ is an embedding. Injectivity of $\tilde{f}_v$ is equivalent to the statement: for all $x \neq y$ and all $\lambda \in \mathbb{R}$, $f(x) - f(y) \neq \lambda v$. Injectivity of $d\tilde{f}_v(p) : T_pM \to v^\perp$ is equivalent to $df(p)w \neq \lambda v$ for all $(p, w) \in TM$ and all $\lambda \in \mathbb{R}$. Thus consider the following two maps from the hint

$$F : M \times M \times \mathbb{R} \to \mathbb{R}^N, \qquad F(p, q, t) = t(f(p) - f(q)),$$

$$G : TM \times \mathbb{R} \to \mathbb{R}^N, \qquad G(p, w, t) = t df(p)w.$$

Since M is compact, the map $\tilde{f}_v$ is always proper. Moreover, it is an embedding if and only if $v \in S^{N-1} \setminus (\operatorname{Im}(F) \cup \operatorname{Im}(G))$.

For $N > 2k+1$ the derivatives of F and G can never be surjective. Hence $\operatorname{Im}(F)$ and $\operatorname{Im}(G)$ are the set of critical value for F and G. Therefore the measure of $\operatorname{Im}(F) \cup \operatorname{Im}(G)$ is zero by Sard's theorem. Hence there exists $0 \neq \xi \in \mathbb{R}^N \setminus (\operatorname{Im}(F) \cup \operatorname{Im}(G)) \neq \emptyset$. Then also $v = \frac{\xi}{\|\xi\|} \in S^{N-1} \setminus (\operatorname{Im}(F) \cup \operatorname{Im}(G))$ an so $\tilde{f}_v : M \to v^\perp \cong \mathbb{R}^{N-1}$ is an embedding.

4. Let $M = S^{k_1} \times \cdots \times S^{k_\ell}$ be a product of spheres. This a smooth manifold of dimension $m = k_1 + \cdots + k_\ell$. Show that M can be embedded into $\mathbb{R}^{m+1}$.

Hint: Prove the following (stronger) statement by induction over ℓ: There exists an embedding $S^{k_1} \times S^{k_2} \cdots \times S^{k_\ell} \times \mathbb{R} \hookrightarrow \mathbb{R}^{m+1}$ where $m = k_1 + \cdots + k_\ell$ as before.

Solution: For $\ell = 1$, the sphere $S^k \subset \mathbb{R}^{k+1}$ is naturally embedded as the unit sphere. We extend this radially to an embedding of $S^k \times \mathbb{R} \hookrightarrow \mathbb{R}^{k+1}$ by the formula

$$f_1 : S^k \times \mathbb{R} \to \mathbb{R}^{k+1}, \qquad f(p, t) := e^t p.$$

Let $M := S^{k_1} \times \cdots \times S^{k_\ell}$ be given and assume by induction that there exists an embedding

$$f_{\ell-1} : N \times \mathbb{R} \to \mathbb{R}^{k_1+\cdots+k_{\ell-1}+1}$$

with $N = S^{k_1} \times \cdots \times S^{k_{\ell-1}}$.

Then we get an embedding $f_\ell : M \times \mathbb{R} \to \mathbb{R}^{k_1+\cdots+k_{\ell-1}+1}$ by the composition of

$$(p, q, t) \in N \times S^{k_\ell} \times \mathbb{R} \mapsto (f_\ell(p, t), q) \in \mathbb{R}^{k_1+\cdots+k_{\ell-1}+1} \times S^{k_\ell}$$

with the map

$$(x, t, q) \in \mathbb{R}^{k_1+\cdots+k_{\ell-1}} \times \mathbb{R} \times S^{k_\ell} \mapsto (x, e^t q) \in \mathbb{R}^{k_1+\cdots+k_{\ell-1}+k_\ell+1}.$$

This proves the statement of the hint. The statement from the exercise follows directly from this by first embedding M as $M \times \{0\}$ into $M \times \mathbb{R}$ and then applying the statement from the hint.

5. Define $f : \mathbb{R}P^2 \to \mathbb{R}^4$ by

$$f([x : y : z]) = \frac{1}{x^2 + y^2 + z^2}(x^2 - y^2, xy, xz, yz)$$

a) Show that f is injective

b) Show that f is an immersion

c) Show that f induces an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$.

Solution:

a) We identify $\mathbb{R}P^2$ with the quotient $S^2/\{\pm 1\}$ of S^2 which is obtained by identifying antipodal points. The given map f naturally lifts to the map

$$F : S^2 \to \mathbb{R}^4, \qquad F(x, y, z) = (x^2 - y^2, xy, xz, yz)$$

and we need to show

$$F(x, y, z) = F(x', y', z') \qquad \Longleftrightarrow \qquad (x, y, z) = \pm(x', y', z').$$

Denote by $(a, b, c, d) = F(x, y, z) = F(x', y', z')$ the image point.

Assume first $a \neq 0$: Then it follows that

$$z^2 = \frac{c^2 + d^2}{a} = (z')^2$$

and hence $z = \pm z'$. When $z \neq 0$, then it follows that $x = c/z = \pm x'$ and $y = d/z = \pm y'$ and the claim follows. When $z, z' = 0$, then follow $x^2 = (a + 1)/2 = (x')^2$ and hence $x = \pm x'$. When $x \neq 0$, then it follows that $y = b/x = \pm y'$ which again proves the claim. When $x, x' = 0$, we get $z^2 = 1 = (z')^2$ and the claim follows once more.

Conversely, assume $a = 0$: That means $x^2 = y^2$ and $(x')^2 = (y')^2$. When $b = 0$, then $x, y, x', y,' = 0$ and $z^2 = 1 = (z')^2$ verifies the claim. When $b > 0$ then $x = y = \pm\sqrt{b}$ and $z = \pm c/\sqrt{b}$. The same applies to (x', y', z') and thus $(x, y, z) = \pm(x', y', z')$. The same argument applies when $b < 0$. In that case follows $x = -y = \pm\sqrt{-b}$ and $z = \pm c/\sqrt{-b}$.

b) In the chart

$$\phi_0 : U_0 := \{[x:y:z] \,|\, x \neq 0\} \to \mathbb{R}^2, \qquad [x:y:z] \mapsto \left(\frac{y}{x}, \frac{z}{x}\right)$$

the the map f is given by

$$f \circ \phi_0^{-1} : \mathbb{R}^2 \to \mathbb{R}^4, \qquad (f \circ \phi_0^{-1})(u,v) = \frac{1}{1+u^2+v^2} \begin{pmatrix} 1-u^2 \\ u \\ v \\ uv \end{pmatrix}.$$

Its derivative is represented by the Jacobian

$$d(f \circ \phi_0^{-1})(u,v) = \frac{1}{(1+u^2+v^2)^2} \begin{pmatrix} -2u(2+v^2) & -2v(1-u^2) \\ 1-u^2+v^2 & -2uv \\ -2uv & 1+u^2-v^2 \\ (1-u^2+v^2)v & (1+u^2-v^2)u \end{pmatrix}.$$

We need to show that this matrix has always rank 2. We investigate first the middle two rows

$$\det \begin{pmatrix} 1-u^2+v^2 & -2uv \\ -2uv & 1+u^2-v^2 \end{pmatrix} = 1-(u^2-v^2)^2 - 4u^2v^2 = 1-(u^2+v^2)^2.$$

It follows that these two rows are linearly independent unless $u^2+v^2=1$. In the case $u^2+v^2=1$ the differential simplifies considerably and we get

$$d(f \circ \phi_0^{-1})(u,v) = \frac{1}{2} \begin{pmatrix} u(2+v^2) & -v^3 \\ v^2 & -uv \\ -uv & u^2 \\ v^3 & u^3 \end{pmatrix} \qquad (\text{when } u^2+v^2=1).$$

A vector $(s,t) \in \mathbb{R}^2$ lies in the kernel of the second or third equation if and only $(s,t) \in \mathbb{R}(u,-v)$. Such a vector is also contained in the kernel of the last equation when $u=0$, $v=0$ or $u^2=v^2$. In each of these cases the vector (s,t) is not in the kernel of the first equation. It follows that $d(f \circ \phi_0^{-1})(u,v)$ has no kernel and hence rank 2.

One can check similarly for the other charts that $d(f \circ \phi_1)$ and $d(f \circ \phi_2)$ always have rank 2. However, we only need to investigate those points which were not already covered. In the chart ϕ_1 we have

$$(f \circ \phi_1^{-1})(u,v) = \frac{1}{1+u^2+v^2} \begin{pmatrix} u^2-1 \\ u \\ uv \\ v \end{pmatrix}$$

and

$$d(f \circ \phi_1^{-1})(u,v) = \frac{1}{(1+u^2+v^2)^2} \begin{pmatrix} 2u(2+v^2) & -2v(u^2-1) \\ 1-u^2+v^2 & -2uv \\ v(1-u^2+v^2) & u(1+u^2-v^2) \\ -2uv & 1+u^2-v^2 \end{pmatrix}.$$

We need to check that $d(f \circ \phi_1^{-1})(0, v) = 0$ has rank 2 for all $v \in \mathbb{R}$. (For all other points $(u, v) \in \mathbb{R}^2$ with $u \neq 0$ it holds $\phi_1^{-1}(u, v) \in U_0$. Differentiating $f \circ \phi_1^{-1} = (f \circ \phi_0^{-1}) \circ (\phi_0 \circ \phi_1^{-1})$ using the chain rule and our previous calculation show then that this derivative is surjective.) In the case $u = 0$ the derivative is given by

$$d(f \circ \phi_1)(0, v) = \frac{1}{(1 + v^2)^2} \begin{pmatrix} 0 & 2v \\ 1 + v^2 & 0 \\ v(1 + v^2) & 0 \\ 0 & 1 - v^2 \end{pmatrix}$$

and this has clearly rank 2.

Finally, it only remains to check that $d(f \circ \phi_3^{-1})(0, 0)$ has rank 2, since all other points correspond to points which are either in U_0 or U_1. We leave this last step to the reader.

c) Since S^2 is compact and the projection map $S^2 \to \mathbb{R}P^2$ is continuous and surjective, it follows that $\mathbb{R}P^2$ is also compact. (Recall from the first exercise sheet that the intrinsic manifold topology on $\mathbb{R}P^2$ agrees with the quotient topology). It follows that $f : \mathbb{R}P^2 \to \mathbb{R}^4$ is proper. By parts (a) and (b) f is also an injective immersion and hence an embedding.

We saw last semester that for an embedding $g : M \to N$ the image $g(M) \subset N$ is a submanifold. The same theorem remains valid in the intrinsic setting with essentially the same proof. In particular, $f(\mathbb{R}P^2)$ is a submanifold of $\mathbb{R}^4$.

6. **a)** Let M_1 be a manifold with boundary and let M_2 be a manifold without boundary. Prove that $M_1 \times M_2$ is a manifold with boundary.

b) If M_1 and M_2 are manifolds with boundary, is $M_1 \times M_2$ a manifold with boundary?

c) Prove that if $\partial M = \emptyset$, then M is a boundary, i.e. there exist W with $\partial W = M$.

d) Let M be a manifold without boundary. Prove that if $a < b$ are two regular values of $f : M \to \mathbb{R}$, then $N = f^{-1}([a, b])$ is a manifold with boundary.

e) Let $P := \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1, 0 \leq x \leq 2\}$ which is a portion of a torus and called pair of pants. Prove that this is a manifold with boundary.

Solution:

a) We need to show that $M_1 \times M_2$ has a canonical structure of a smooth manifold with boundary. This is obtained as follows: Let $\mathcal{A} = \{\phi_i : U_i \to \mathbb{H}^{m_1}\}_{i \in A}$ be an atlas of M_1 and let $\mathcal{B} = \{\psi_j : V_j \to \mathbb{R}^{m_2}\}_{j \in B}$ be an atlas of M_2. For $i \in A$ and $j \in B$, we get the product maps

$$\phi_i \times \psi_j : U_i \times V_j \to \mathbb{H}^{m_1} \times \mathbb{R}^{m_2} \cong \mathbb{H}^{m_1 + m_2}$$

combine to an atlas of $M_1 \times M_2$. These maps are homeomorphism in the product topology and the transition maps are clearly smooth.

b) The construction above does not carry over when M_2 is also a manifold with boundary, since $\mathbb{H}^{m_1} \times \mathbb{H}^{m_2}$ is not diffeomorphic to an open subset of $\mathbb{H}^{m_1 + m_2}$. The product space is called a manifolds with corners.

c) The boundary of $W := M \times [0, 1)$ is $M \times \{0\}$ and thus diffeomorphic to M.

A much more difficult question is for which M exists a compact manifold W with $\partial W = M$. Such a manifold M is called null-cobordant. The simplest example of a manifold which is not null-cobordant is $\mathbb{R}P^2$. (The proof of this fact is by no means obvious!)

d) We have seen in the lecture that

$$M_a := \{x \in M \mid f(x) \geq a\} = f^{-1}([a, \infty)),$$
$$M_b := \{x \in M \mid f(x) \leq b\} = f^{-1}((-\infty, b])$$

are smooth manifolds with boundary, provided that $a, b \in \mathbb{R}$ are regular values for f. The intersection of these two manifolds is given by

$$N := f^{-1}([a, b]) = M_a \cap M_b.$$

For any point $p \in N$ there exists an open neighbourhood $p \in U_p \subset N$ (open in the relative topology of N) such that $U_p \subset M_a$ or $U_p \subset M_b$. We can thus cover N by charts, which are obtained from restricting charts of M_a or M_b. Those charts are smooth diffeomorphism with respect to the smooth manifold structure of M and therefore smoothness of the transition maps is automatic.

e) The torus

$$M := \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\} \subset \mathbb{R}^3$$

is a smooth submanifold of $\mathbb{R}^3$ and

$$P := f^{-1}([0, 2])$$

for $f : M \to \mathbb{R}, \qquad f(x, y, z) = x$. We use the general fact that the restriction of a smooth function on a manifold to a submanifold is again smooth and the differential of the restriction is obtained by restricting the differential of the function to the manifold. In our case, it follows that f is smooth with differential

$$df(x, y, z) := T_{(x,y,z)}M \to \mathbb{R}, \qquad (\hat{x}, \hat{y}, \hat{z}) \mapsto \hat{x}$$

since the function $\mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto x$ is clearly smooth with the same differential.

We need to show that 0 and 2 are regular values for f. By the formula above, we need to show that $T_{(x,y,z)}M$ is not the (y, z)-plane for $(x, y, z) \in f^{-1}(0) \cup f^{-1}(2)$. Since M is a level set of the function

$$h : \mathbb{R}^3 \to \mathbb{R}, \qquad h(x, y, z) = \sqrt{x^2 + y^2} - 2)^2 + z^2$$

the tangent space $T_{(x,y,z)}M$ agrees with the orthogonal complement of

$$\nabla h(x, y, z) = 2 \begin{pmatrix} x(\sqrt{x^2 + y^2} - 2)/\sqrt{x^2 + y^2} \\ y(\sqrt{x^2 + y^2} - 2)/\sqrt{x^2 + y^2} \\ z \end{pmatrix}.$$

This is vector is not parallel to the the x-axis for $(x, y, z) \in f^{-1}(0) \cup f^{-1}(2)$ and this proves the claim. It now follows from part (c) that P is a manifold with boundary.