Solution 3

1. Let $f : S^1 \to S^1$ be a smooth map.

a) Show that there exists $k \in \mathbb{Z}$ and a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ with

$$\phi(x+1) = \phi(x) + k \qquad \text{and} \qquad f(\exp(2\pi \mathbf{i} x)) = \exp(2\pi \mathbf{i}\phi(x))$$

for all $x \in \mathbb{R}$.

b) Show that $\deg(f) = k$, where $k \in \mathbb{Z}$ is as in part (a) .

Hint: For b): Show first that f is homotopic to $z \mapsto z^k$. For this part a) might come in handy.

Solution:

a) Our solution uses some standard results for covering maps from point-set topology. The map

$$p : \mathbb{R} \to S^1, \qquad p(x) = \exp(2\pi \mathbf{i} x)$$

is a smooth covering map. It follows from the path-lifting property of covering spaces, that the path

$$\gamma(x) : \mathbb{R} \to S^1, \qquad \gamma(x) := f(p(x)) = f(\exp(2\pi \mathbf{i} x))$$

has a continuous lift $\phi : \mathbb{R} \to \mathbb{R}$ satisfying

$$p \circ \phi(x) = \gamma(x) \qquad \text{for all } x \in \mathbb{R}.$$

Since $p(x+1) = p(x)$, it follows that $\gamma(x+1) = \gamma(x)$ and hence $p(\phi(x)) = p(\phi(x+1))$. This implies $\phi(x+1) - \phi(x) \in \mathbb{Z}$ for all $x \in \mathbb{R}$. Since $x \mapsto \phi(x+1) - \phi(x)$ is continuous and $\mathbb{Z}$ is totally disconnected, it follows that $\phi(x+1) - \phi(x) \equiv k \in \mathbb{Z}$ is constant.

It is not hard to see that the lift of a smooth function under a smooth covering map is always smooth. In our case at hand this observation becomes somewhat tautological: For open interval $I \subset \mathbb{R}$ of length $|I| < 1$ we have a smooth parametrization

$$\psi_I : I \to p(I) \subset S^1, \qquad \psi_I(x) = p(x)$$

and its inverse map $\varphi_I := \psi_I^{-1} : p(I) \to I$ defines a smooth chart for S^1. The collection of these charts defines an atlas of S^1. If we express f in these coordinates, we get

$$(\varphi_I \circ f \circ \psi_J)(x) := \phi(x) + r$$

for some $r \in \mathbb{Z}$ only depending on the charts and not the variable x. Therefore f is smooth if and only if ϕ is smooth.

b) For $0 \le t \le 1$ define

$$\phi_t : \mathbb{R} \to \mathbb{R}, \qquad \phi_t(x) = t\phi(x) + (1-t)kx.$$

This satisfies $\phi_t(x+1) = \phi_t(x) + k$ and hence determines a smooth function

$$f_t : S^1 \to S^1, \qquad f_t(\exp(2\pi \mathbf{i} x)) = \exp(2\pi \mathbf{i}\phi_t(x)).$$

Now $\{f_t\}$ is a smooth homotopy with $f_1 = f$ and $f_0(\exp(2\pi \mathbf{i} x)) = \exp(2\pi \mathbf{i} k x)$, i.e. $f_0(z) = z^k$. For $k = 0$, this function is constant and has degree 0. For $k \neq 0$ the values $1 \in S^1$ is a regular value of f_0 with $|k|$ preimages: the k-th roots of unity or their conjugates depending on the sign of k. Moreover, f_0 is orientation preserving for $k > 0$ and orientation reversing for $k < 0$. Hence

$$\deg(f) = \deg(f_0) = k$$

by homotopy invariance of the degree.

2. **[Existence of a Morse function]** Let $M^m \subset \mathbb{R}^n$ be a smooth manifold and $f : M \to \mathbb{R}$ be a smooth function. The Hessian of f at $p \in M$ is the bilinear map

$$H_p f : T_pM \times T_pM \to \mathbb{R}$$

defined by the covariant derivative of df, i.e.

$$H_p f(X, Y) := \nabla_X (df)_p(Y) := \mathcal{L}_X(df(p)Y) - df(p)\nabla_X Y.$$

This is always symmetric and does not depend on the embedding when $df(p) = 0$ is a critical point. We call f a **Morse function**, when $H_p f$ is nondegenerate for all $p \in M$ with $df(p) = 0$.

Denote by $TM^\perp := \{(p, v) \mid p \in M,\, v \in T_pM^\perp\}$ the normal bundle of M and define

$$\phi : TM^\perp \to \mathbb{R}^n, \qquad \phi(p, v) = p + v.$$

a) Prove that $x \in \mathbb{R}^n$ is a regular value of ϕ if and only if the function

$$f_x : M \to \mathbb{R}, \qquad f_x(p) := \frac{1}{2}\|p - x\|^2$$

is a Morse function on M.

b) Prove that there exists a Morse function on every manifold M.

Remark: Morse functions are an important tool in differential topology: They are used to construct various invariants of manifolds, to classify closed 2-manifolds and to prove the famous h-cobordism theorem in higher dimensions.

Solution:

a)

- First of all, let us see under which condition $p \in M$ is a critical point of f_x. The derivative of f at $p \in M$ is

$$df_x(p) : T_pM \to \mathbb{R}, \qquad df_x(p)\hat{p} = \langle p - x, \hat{p} \rangle.$$

Thus p is a critical point of f_x if and only if $p - x \in T_pM^\perp$.

- The Hessian of f_x is given by

$$\begin{aligned} H_p f_x(X, Y) &= \mathcal{L}_X \langle p - x, Y(p) \rangle - \langle p - x, \nabla_X Y(p) \rangle \\ &= \langle X(p), Y(p) \rangle + \langle p - x, dY(p)X(p) \rangle - \langle p - x, \nabla_X Y(p) \rangle \\ &= \langle X(p), Y(p) \rangle + \langle p - x, h_p(X(p), Y(p)) \rangle \end{aligned}$$

where $h_p : T_pM \times T_pM \to T_pM^\perp$ denotes the second fundamental form. It is characterised by the equation $dY(p)X(p) = \nabla_X Y(p) + h_p(X(p), Y(p))$.

It follows that f_x is a Morse function on M if and only if the bilinear form

$$H_p f_x : T_pM \times T_pM \to \mathbb{R}, \qquad H_p f_x(X, Y) := \langle X, Y \rangle + \langle p - x, h_p(X, Y) \rangle$$

is nondegenerate for all $p \in M$ with $p - x \in T_pM^\perp$.

- We now have a look at the function $\phi : TM^\perp \to \mathbb{R}^n$. It is the restriction of a linear function and so its derivative is given by

$$d\phi(p,v) : T_{(p,v)}(TM^\perp) \to \mathbb{R}^n, \qquad d\phi(p,v)(\hat{p},\hat{v}) = \hat{p} + \hat{v}.$$

Denote by $\Pi(p) : \mathbb{R}^n \to T_pM$ the smooth family of orthogonal projections. Then $TM^\perp$ is the submanifold of $M \times \mathbb{R}^n$ traced out by the equation $\Pi(p)v = 0$ and so its tangent space is given by

$$T_{(p,v)}(TM^\perp) = \{(\hat{p},\hat{v}) \in T_pM \times \mathbb{R}^n \,|\, (d\Pi(p)\hat{p})v + \Pi(p)\hat{v} = 0\}.$$

Recall that $(d\Pi(p)\hat{p})v = h_p(\hat{p})^*v$ is the adjoint of the second fundamental form.

We can now verify regular values x of ϕ give Morse functions f_x.

For this, suppose $\phi(p,v) = x$ and take $\xi \in \mathbb{R}^n$ such that

$$\langle \xi, d\phi(p,v)(\hat{p},\hat{v})\rangle = \langle \xi, \hat{p} + \hat{v}\rangle = 0 \qquad \text{for all } (\hat{p},\hat{v}) \in T_{(p,v)}(TM^\perp).$$

Since $(0,\hat{v}) \in T_{(p,v)}(TM^\perp)$ for all $\hat{v} \in T_pM^\perp$, it follows that $\xi \in T_pM$. Using the equation $h_p(\hat{p})^*v + \Pi(p)\hat{v} = 0$, we obtain

$$0 = \langle \xi, \hat{p} + \hat{v}\rangle = \langle \xi, \hat{p}\rangle - \langle \xi, h_p(\hat{p})^*v\rangle = \langle \xi, \hat{p}\rangle - \langle h_p(\xi,\hat{p}), v\rangle = \langle \xi, \hat{p}\rangle + \langle h_p(\xi,\hat{p}), p - x\rangle$$

for all $T_{(p,v)}(TM^\perp)$. The projection $T_{(p,v)}(TM^\perp) \to T_pM : (\hat{p},\hat{v}) \mapsto \hat{p}$ is clearly surjective (as $TM^\perp$ is a vector bundle and the canonical projection a submersion). It follows that $d\phi(p,v)$ is surjective if and only if

$$0 = \langle \xi, \hat{p}\rangle + \langle h_p(\xi,\hat{p}), p - x\rangle = H_pf_x(\xi,\hat{p}) \quad \forall\, \hat{p} \in T_pM \qquad \Longrightarrow \qquad \xi = 0$$

where $v = x - p \in T_pM^\perp$. By our calculation above, this is equivalent to f_x being a Morse function.

b) By Whitney embedding, we can always embed $M \subset \mathbb{R}^n$. By a) and Sard's Theorem, there exists a regular value x of ϕ. Thus f_x is a Morse function on M.

3. Let $f : M \to N$ and $g : N \to Q$ be smooth maps between m-dimensional compact connected manifolds. Show that $\deg(g \circ f) = \deg(f) \cdot \deg(g)$.

Solution: Let z be a regular value for $g \circ f$. Denote is preimages under g by

$$g^{-1}(z) := \{y_1, \ldots, y_\ell\}$$

We group the preimages under $(g \circ f)$ into the sets

$$f^{-1}(y_j) = \{x_1^{(j)}, \ldots, x_{r_j}^{(j)}\}$$

for $j = 1, \ldots, \ell$. Since z is a regular value of $g \circ f$, it follows that the derivatives

$$d(g \circ f)(x_i^{(j)}) = dg(y_j) \circ df(x_i^{(j)})$$

are bijective and in particular, $dg(y_j)$ is surjective and $df(x_i^{(j)})$ injective. Since they are linear maps between vector spaces of the same dimension, they must both be bijective. It follows that the preimages y_j are regular values for f.

Define $\epsilon_i^{(j)} = +1$ when $df(x_i^{(j)})$ is orientation preserving and $\epsilon_i^{(j)} = -1$ otherwise. Similarly define $\epsilon_j = +1$ when $dg(y_j)$ is orientation preserving and $\epsilon_j = -1$ otherwise. Then

$$\deg(g) = \sum_{j=1}^{\ell} \epsilon_j, \qquad \deg(f) = \sum_{i=1}^{r_j} \epsilon_i^{(j)}$$

We have seen in the lecture that the last sum does not depend on the regular value y_j chosen for the computation of $\deg(f)$.

Finally, $d(g \circ f)(x_i^{(j)})$ is orientation preserving if and only if $dg(y^j)$ and $df(x_i^{(j)})$ are both orientation preserving or both reversing. The corresponding sign is given by $\epsilon_j \epsilon_i^{(j)}$ and thus

$$\deg(g \circ f) = \sum_{j=1}^{\ell} \sum_{i=1}^{r_j} \epsilon_j \epsilon_i^{(j)} = \sum_{j=1}^{\ell} \epsilon_j \deg(f) = \deg(f) \cdot \deg(g).$$

4. Show that every smooth map $f : S^n \to S^n$ of degree different from $(-1)^{n+1}$ must have a fixed point.

Hint: Suppose f has no fixed point. Then there exists a homotopy between f and the antipodal map $S^n \to S^n, x \mapsto -x$.

Solution: Suppose f has no fixed point. Then for every $x \in S^n$ the line passing through $f(x)$ and $-x$ does not contain the origin. Hence the following homotopy is well-defined

$$[0,1] \times S^n \to S^n, \qquad (t, x) \mapsto f_t(x) := \frac{tf(x) - (1-t)x}{\|tf(x) - (1-t)x\|}$$

This satisfies $f_1 = f$ and $f_0(x) = -x$ is the antipodal map. The degree of the antipodal map is $(-1)^{n+1}$ and by homotopy invariance it follows

$$\deg(f) = \deg(f_0) = (-1)^{n+1}.$$

Conversely, it follows that f has a fixed point whenever its degree is different from $(-1)^{n+1}$.

5. **a)** Suppose $g : S^n \to S^n$ satisfies $g(x) = g(-x)$ for all $x \in S^n$. Then the degree of g is even.

b) Suppose $f : S^n \to S^n$ is a smooth function with odd degree. Then f carries some pair of antipodal points into a pair of antipodal points.

Solution:

a) The degree of g is defined as the signed count of the preimages $g^{-1}(y)$ of some regular value y. Since these preimages come as pairs of antipodal points, the total number of preimages must be even and hence their signed count is even. (Note that we do not need to know which signs are $+1$ and which are -1 for this argument!)

b) Suppose f carries no pair of antipodal points into a pair of antipodal points. This means that for every $x \in S^n$ the line through $f(x)$ and $f(-x)$ does not contain the origin. Then the following homotopy is well-defined

$$[0,1] \times S^n \to S^n, \qquad \frac{tf(x) + (1-t)f(-x)}{\|tf(x) + (1-t)f(-x)\|}$$

Then $f_1(x) = f(x)$, and $f_{1/2}(x) = \frac{1}{2}(f(x) + f(-x)) = f_{1/2}(-x)$. It follows

$$\deg_2(f) = \deg_2(f_{1/2}) = 0$$

by part (a) and the homotopy invariance of the degree.

Conversely, whenever f has odd degree, it must carry some pair of antipodal points into a pair of antipodal points.

6. **a)** Show that the tangent bundle TM of a smooth manifold M is always an orientable manifold.

b) * Show that every simply connected, connected manifold M is orientable.

Hint: For a): Have a closer look at the atlas for TM discussed in Exercise Sheet 1, Exercise 5.
For b): Fix $p \in M$ and a frame $e : \mathbb{R}^m \to T_pM$. For any point $q \in M$ choose a smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$ and choose a smooth lift $\beta(t) = (\gamma(t), e(t))$ in the frame bundle. We decree that $e(1) : R^m \to T_qM$ is orientation preserving. Now verify (1) this does not depend on the choices made for γ and β and (2) these orientations on the tangent spaces T_qM fit "fit smoothly together" and define an orientation on M.

Solution:

a) We need to find an atlas for TM with orientation preserving transition maps. We verify that the canonical atlas for TM, obtained from any atlas for M which itself might not be orientable, has always orientation preserving transition maps.

Let $\mathcal{A} := \{\phi_\alpha, U_\alpha\}_{\alpha\in A}$ be an atlas for M. Denote by $\pi : TM \to M$ the canonical projection map. Then

$$\tilde{\phi}_\alpha : \tilde{U}_\alpha := \pi^{-1}(U_\alpha) \to \mathbb{R}^m \times \mathbb{R}^m, \qquad \tilde{\phi}_\alpha(p,v) := (\phi_\alpha(p), d\phi_\alpha(p)v)$$

are charts for TM and the union $\tilde{\mathcal{A}} := \{\tilde{\phi}_\alpha, \tilde{U}_\alpha\}_{\alpha\in A}$ is the induced atlas for TM. Its transition maps are given by

$$\begin{aligned}\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1} : \phi_\beta\left(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)\right) &\to \phi_\alpha\left(\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)\right) \\ (x,\xi) &\mapsto \left((\phi_\alpha \circ \phi_\beta^{-1})(x),\, d(\phi_\alpha \circ \phi_\beta^{-1})(x)\xi\right).\end{aligned}$$

Its derivative is given by

$$d(\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1})(x,\xi) = \begin{pmatrix} d(\phi_\alpha \circ \phi_\beta^{-1})(x) & 0 \\ d^2(\phi_\alpha \circ \phi_\beta^{-1})(x)\xi & d(\phi_\alpha \circ \phi_\beta^{-1})(x) \end{pmatrix}$$

(The second derivative $d^2(\phi_\alpha \circ \phi_\beta^{-1})(x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric bilinear map. Plugging ξ into one of its arguments yields the linear map $d^2(\phi_\alpha \circ \phi_\beta^{-1})(x)\xi$). It follows

$$\det\left(d(\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1})(x,\xi)\right) = \det\left(d(\phi_\alpha \circ \phi_\beta^{-1})(x)\right)^2 > 0$$

and $\tilde{\mathcal{A}}$ is an oriented atlas for TM.

b) Fix $p \in M$ and a frame $e : \mathbb{R}^m \to T_pM$. For $q \in M$ choose a smooth path in the frame bundle

$$\beta : [0,1] \to \mathcal{F}(M), \qquad \beta(t) = (\gamma(t), e(t))$$

where $\gamma : [0,1] \to M$ is a smooth path with $\gamma(0) = p$, $\gamma(1) = q$ and $e(t) : \mathbb{R}^m \to T_{\gamma(t)}M$ is a smooth path of frames along γ starting at $e_0 = e$.

Step 1: *The orientation induced by* $e(1) : \mathbb{R}^m \to T_qM$ *depends only on the path* γ *and the frame* e.

Any other choice for the frames has the shape $\tilde{e}(t) = e(t)g(t)$ for a smooth curve $g : [0,1] \to \mathrm{GL}(m,\mathbb{R})$ with $g(0) = \mathbb{1}$. In particular, $\tilde{e}(1)$ can be smoothly deformed to $e(1)$ through the frames $e(1)g(t)$. Hence they induce the same orientation.

Step 2: *The orientation induced by* $e(1) : \mathbb{R}^m \to T_qM$ *depend only on the pair* (p,e) *and not on the choice of the path* γ.

Let $\gamma_0, \gamma_1 : [0,1] \to M$ be two paths with $\gamma_0(0) = p = \gamma_1(0)$ and $\gamma_0(1) = q = \gamma_1(1)$. Since M is simply connected, these paths are homotopic, i.e. exists a smooth map

$$[0,1] \times [0,1] \to M, \qquad (s,t) \mapsto \gamma_s(t)$$

with $\gamma_s(0) = p$ and $\gamma_s(1) = q$ for all $s \in [0,1]$. Let

$$\beta : [0,1] \times [0,1] \to \mathcal{F}(M), \qquad \beta(s,t) = (\gamma_s(t), e(s,t))$$

be a lift of this homotopy. (One way to see this is using the theory of fiber bundles: The bundle $\mathcal{F}(M) \to M$ is a locally trivial fiber bundle and hence has the homotopy lifting property. Alternatively, choose an embedding $M \subset \mathbb{R}^n$. Use parallel transport to define the lift $\beta(0,t)$ along γ_0. And then use parallel transport again to extend this lift along the paths $s \mapsto \gamma(s,t)$ for every t. These fit together smoothly by smooth dependence of the solution of an ODE on its initial condition.)

It now follows that $s \mapsto e(s,1)$ is a path of frames for T_qM connecting one frame coming from γ_0 to another frame coming from γ_1. Hence Step 2 follows from Step 1.

Step 3: M *is orientable.*

By Step 2, we have a preferred orientation of all tangent spaces T_qM, which only depends on the choice of (p,e). We need to show that these give rise to an oriented atlas. For $q \in M$ choose a chart

$$\phi : U \to \mathbb{R}^m$$

defined on a connected neighborhood of q and such that $d\phi(q) : T_qM \to \mathbb{R}^m$ is orientation preserving. (We can guarantee this by otherwise composing ϕ with a reflection on $\mathbb{R}^m$).

For any point $y \in U$ choose a path $\gamma : [0,1] \to U \subset M$ with $\gamma(0) = q$ and $\gamma(1) = y$. Lift this path to the frame bundle using the frames

$$e(t) := (d\phi(\gamma(t)))^{-1} : \mathbb{R}^m \to T_{\gamma(t)}M.$$

By definition of the preferred orientation on T_yM, it follows that $e(1) : \mathbb{R}^m \to T_yM$ is orientation preserving. In particular, $d\phi(y) : T_yM \to \mathbb{R}^m$ is orientation preserving for all $y \in U$. This is precisely the condition need for pointwise defined orientations to combine to an orientation of M.