Solution 4

1. Let $M^m \subset \mathbb{R}^n$ be a smooth, compact manifold without boundary and for $\epsilon > 0$ denote

$$N_\epsilon := \left\{ p + v \,|\, p \in M,\, v \in T_pM^\perp,\, \|v\| \le \epsilon \right\}.$$

a) Prove that for $\epsilon > 0$ sufficiently small N_ϵ is a smooth manifold with boundary.

b) Prove that there exists a unique smooth map $r : N_\epsilon \to M$ that satisfies

$$\|x - r(x)\| = \min_{p \in M} \|x - p\|.$$

Show for all $x \in N_\epsilon$ that $x - r(x) \perp T_{r(x)}M$.

Hint: For a): Show that $\phi : TM^\perp \to \mathbb{R}^n$, $\phi(p, v) := p + v$ restricts to an embedding on a suitable neighborhood of $M \times \{0\} \subset TM^\perp$. For b): It holds $r(p + v) = v$. Uniqueness of the map r follows from injectivity of ϕ.

Solution:

a) We have seen in the last semester that the normal bundle

$$TM^\perp = \{(p, v) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, p \in M,\, v \in T_pM^\perp\}$$

is a smooth vector bundle and in particular a smooth manifold. Consider the map

$$\phi : TM^\perp \to \mathbb{R}^n, \qquad \phi(p, v) := p + v.$$

For $p \in M$ the derivative of ϕ at the point $(p, 0)$ is given by

$$d\phi(p, 0) : T_{(p,0)}\left(TM^\perp\right) = T_pM \oplus T_pM^\perp \to \mathbb{R}^n, \qquad (\hat{p}, \hat{v}) \mapsto \hat{p} + \hat{v}.$$

See Exercise 2 on Sheet 3 for a discussion of the tangent spaces $T_{(p,v)}\left(TM^\perp\right)$. Since $d\phi(p, 0)$ is bijective, there exists an open neighborhood of $(p, 0)$ in $TM^\perp$ such that ϕ restricts to a diffeomorphism.

We show that there exists $\epsilon_1 > 0$ such that

$$\phi : \{(p, v) \in TM^\perp \,|\, \|v\| < \epsilon_1\} \to \left\{ p + v \,|\, p \in M,\, v \in T_pM^\perp,\, \|v\| < \epsilon_1 \right\}$$

is bijective. We argue by contradiciton: If this fails for all $\epsilon > 0$, then there exist sequences $(p_i, v_i), (q_i, w_i) \in TM^\perp$ such that

$$p_i + v_i = q_i + w_i \quad \forall i \qquad \text{and} \qquad \lim_{i\to\infty} \|v_i\| = \lim_{i\to\infty} \|w_i\| = 0.$$

Up to choosing a subsequence, it follows that $\lim_{i\to\infty} p_i = \lim_{i\to\infty} q_i =: x$ and this contradicts the observation from above, that ϕ restricts to a diffeomorphism on some neighborhood of $(x, 0)$.

Since $d\phi(p, 0)$ is bijective for all $p \in M$, and bijectivity is an open condition, there exists an open neighborhood of $M \subset W \subset TM^\perp$ such that $d\phi(p, v)$ is bijective for all $(p, v) \in W$. Since M is compact, there exists $\epsilon_2 > 0$ such that $\left\{ p + v \,|\, p \in M,\, v \in T_pM^\perp,\, \|v\| < \epsilon_2 \right\} \subset W$. Hence for $\epsilon_3 := \min\{\epsilon_1, \epsilon_2\}$, we have that

$$\phi : \{(p, v) \in TM^\perp \,|\, \|v\| < \epsilon_3\} \to \left\{ p + v \,|\, p \in M,\, v \in T_pM^\perp,\, \|v\| < \epsilon_3 \right\}$$

is a injective immersion. It is also proper, since $\{(p,v) \in TM^\perp \mid |v| < \epsilon_3\}$ is a bounded subset of $\mathbb{R}^n \times \mathbb{R}^n$. It follows that

$$N_{\epsilon_3}^\circ = \Big\{p + v \,|\, p \in M,\, v \in T_pM^\perp,\, \|v\| < \epsilon_3\Big\} \subset \mathbb{R}^3$$

is a smooth manifold.

Finally, the function

$$f : N_{\epsilon_3}^\circ \to \mathbb{R}, \qquad f(p+v) := \|v\|^2$$

is a well-defined smooth function, because ϕ restricts to a diffeomorphism. Scaling of v yields a non-vanishing directional derivative at every point $p + v$ with $v \neq 0$. In particular, every $0 < \epsilon < \epsilon_3$ is a regular value. It then follows that $N_\epsilon := f^{-1}((-\infty, \epsilon])$ is a smooth manifold with boundary.

b) Fix $x \in \mathbb{R}^n$ and the function

$$f_x : M \to \mathbb{R}, \qquad f_x(p) := \frac{1}{2}\|p - x\|^2$$

is a smooth function with derivative

$$df_x(p) : T_pM \to \mathbb{R}, \qquad df_x(p)\hat{p} := \langle p - x, \hat{p}\rangle.$$

In particular, $p \in M$ is a critical point of f_x if and only if $p - x \in T_pM^\perp$.

It follows from part a) that every point $x \in N_\epsilon$ can uniquely be expressed as $x = p+v$ with $v \in T_pM^\perp$. We claim that p is the unique global minimum of f_x. Suppose otherwise that there exists $p' \in M$ such that

$$\frac{1}{2}\|v\| = f(p) \geq f(p') = \min\nolimits_{q\in M} f_x(q).$$

This point is necessarily a critical point of f_x and thus $v' := x - p' \in T_{p'}M^\perp$. It follows that $p+v = x = p'+v'$ and, since $\|v'\| \leq \|v\| \leq \epsilon$, this contradicts injectivity of ϕ unless $p = p'$.

We have thus shown that $r(x) = r(p + v) = p$. Under the diffeomorphism ϕ, this map r corresponds to the canonical projection $TM^\perp \to M$ of the normal bundle and therefore it is smooth.

2. Let M be a m-dimensional oriented compact manifold with boundary and let $f : M \to \mathbb{R}^m$ be a smooth map. For a regular value $y \in \mathbb{R}^m \setminus f(\partial M)$ define

$$g_y : \partial M \to S^{m-1}, \qquad g_y(p) := \frac{f(p) - y}{\|f(p) - y\|}.$$

Prove that

$$\deg(g_y) = \deg(f, y) = \sum_{p\in f^{-1}(y)} \operatorname{sign}(df(p)).$$

Hint: The modulo 2 version of this result has been proven in Chapter VI, Lemma 7 of the lecture. The same argument works in general, but you have to keep track of the orientations.

Solution: We remind the reader that the degree $\deg(f;y)$ usually depends on the regular value y and might change when we modify f to some homotopic function $\tilde{f}$. This is the case because f is defined on a manifold with boundary (and things might go wrong when y crosses $f(\partial M)$). The simplest example to envision is the embedding of the closed unit ball into $\mathbb{R}^n$.

Now let $y \in \mathbb{R}^n \setminus f(\partial M)$ be a regular value of f with preimages

$$f^{-1}(y) := \{p_1, \dots, p_k\} \subset M.$$

For each preimage take an open neighborhood $p_j \in U_j \subset M$ and a chart

$$\phi_j : U_k \to \Omega_k \subset \mathbb{R}^m$$

which sends $\phi_k(p_j) = 0 \in \Omega_k$. Then choose $\epsilon_j > 0$ sufficiently small such that $B_{\epsilon_j}(0) \subset \Omega_j$ and the closures of

$$V_j := \phi_j^{-1}(B_{\epsilon_j}(0)) \subset M$$

do not intersect ∂M and are pairwise disjoint. Then

$$N := M \setminus (V_1 \cup \ldots \cup V_k)$$

is a smooth manifold with boundary. The function g_y extends over N to the smooth function

$$G_y : N \to S^{m-1}, \qquad g_y(p) := \frac{f(p) - y}{\|f(p) - y\|}.$$

and therefore $\deg\left(G_y|_{\partial N} : \partial_N \to S^{m-1}\right) = 0$. The boundary of N is the disjoint ∂M and ∂V_j for $j = 1, \dots, k$. It follows that

$$\deg(g_y) = \deg\left(G_y|_{\partial M} : \partial M \to S^{m-1}\right) = \sum_{j=1}^{k} \deg\left(G_y|_{\partial V_j} : \partial V_j \to S^{m-1}\right)$$

In the last equation, we view ∂V_j as the boundary of V_j. This yields the opposite orientation as ∂N (and an extra sign in the calculation).

Define

$$h_j : S^{m-1} \to S^{m-1}, \qquad h_j(x) := G_y(\phi_j^{-1}(\epsilon_j x)) = \frac{f(\phi_j^{-1}(\epsilon_j x)) - y}{\|f(\phi_j^{-1}(\epsilon_j x)) - y\|}.$$

Since ϕ_j is a chart, it restricts to a diffeomorphism of $\partial B_{\epsilon_j}(0)$ to ∂V_j. By the composition formula for the degree (see Exercise 3 on Sheet 3) it follows

$$\deg\left(G_y|_{\partial V_j} : \partial V_j \to S^{m-1}\right) = \deg(h_j).$$

It remains to show that the degree of h_j agrees with the sign of $df(p_j)$. This follows from a homotopy argument. Define

$$h_{j,t}(x) = \frac{f(\phi_j^{-1}(t\epsilon_j x)) - y}{\|f(\phi_j^{-1}(t\epsilon_j x)) - y\|} \qquad \text{for } 0 < t \leq 1,$$

$$h_{j,0}(x) = \frac{d\left(f \circ \phi_j^{-1}\right)(0)x}{\|d\left(f \circ \phi_j^{-1}\right)(0)x\|} \qquad \text{for } t = 0.$$

Then $[0,1] \times S^{m-1} \to S^{m-1}$, $(t,x) \mapsto h_{j,0}(x)$, is a smooth homotopy, since $f \circ \phi_j^{-1}$ is a smooth function and $d(f \circ \phi_j^{-1})(0)$ is an isomorphism according to the assumption that p_j is a regular point for f. Hence

$$\deg(h_j) = \deg(h_{j,0}).$$

Now as a general fact, when $L : \mathbb{R}^m \to \mathbb{R}^m$ is a linear isomorphism, then

$$\deg\left(\lambda_L : S^{m-1} \to S^{m-1}, \qquad \lambda_L(x) \mapsto \frac{Lx}{\|Lx\|}\right) = \mathrm{sign}(\det(L)).$$

One way to see this is to choose a smooth path of invertible matrices L_t with $L_1 = L$ and $L_0 = \mathbb{1}$ when $\det(L) > 0$ and $L_0 = \mathrm{diag}(-1, 1, \ldots, 1)$ when $\det(L) < 0$. The claim is readily checked for L_0 and then follows by homotopy invariance of the degree for $L = L_1$. With $L = d\left(f \circ \phi_j^{-1}\right)(0)$, we finally conclude

$$\deg(h_{j,0}) = \mathrm{sign}\left(\det(d\left(f \circ \phi_j^{-1}\right)(0))\right)$$

and this is the sign associated to p_j when calculating the degree of f.

3. Given disjoint manifolds $M, N \subset \mathbb{R}^{k+1}$, the linking map is defined by

$$\lambda : M \times N \to S^k, \qquad \lambda(x,y) := \frac{x-y}{\|x-y\|}.$$

If M^m and N^n are compact, oriented, and boundaryless, with total dimension $m + n = k$, then the degree of λ is called the *linking number* $\ell(M,N)$.

a) Prove that $\ell(N,M) = (-1)^{(m+1)(n+1)}\ell(M,N)$.

b) If M bounds an oriented manifold X disjoint from N, then $\ell(M,N) = 0$.

c) Define the linking number for disjoint submanifolds $M^m, N^n \subset S^{k+1}$ for $m, n \geq 1$.

Hint: For c): You may use the fact that $S^{m+n+1} \backslash (N \cup M)$ is nonempty and path-connected for $m, n \geq 1$ without proof.

Solution:

a) Let $\lambda : M \times N \to S^k$ and $\mu : N \times M \to S^k$ be the linking maps defined for the pair (M,N) and (N,M) defined by

$$\lambda(x,y) = \frac{x-y}{\|x-y\|}, \qquad \mu(y,x) = \frac{y-x}{\|y-x\|}$$

Then $\mu = \tau \circ \lambda \circ s$ where $s : M \times N \to N \times M$ is the swapping map defined by $s(p,q) := (q,p)$ and $\tau : S^k \to S^k$ is the antipodal map defined by $\tau(p) = -p$. The composition formula for the degree yields:

$$\ell(N,M) = \deg(\mu) = \deg(\tau \circ \lambda \circ s) = \deg(\tau)\deg(s)\deg(\lambda) = \deg(\tau)\deg(s)\ell(M,N).$$

The degree of the antipodal map has been calculated elsewhere, it is given by $\deg(\tau) = (-1)^{k+1}$. The swapping map s has degree $(-1)^{mn}$. To see this, take oriented atlases $\{(U_i \subset M, \varphi_i)\}_{i\in A}$ and $\{(V_j \subset N, \varphi_j)\}_{j\in B}$, and express s in the charts as

$$s_{ij}(x,y) := (\psi_j \times \varphi_i) \circ s \circ (\varphi_i^{-1} \times \psi_j^{-1})(x,y) = (y,x).$$

for $(i,j) \in A \times B$. This is clearly a diffeomorphism and the determinant of s_{ij} is equal to

$$\det \begin{pmatrix} 0 & \mathbb{1}_m \\ \mathbb{1}_n & 0 \end{pmatrix} = (-1)^n \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathbb{1}_{m-1} \\ 0 & \mathbb{1}_n & 0 \end{pmatrix} = \ldots = (-1)^{nm} \det(\mathbb{1}_{m+n}) = (-1)^{nm}.$$

It follows that

$$\ell(N,M) = (-1)^{nm+n+m+1}\ell(M,N).$$

b) When $M = \partial X$ with $X \cap N = \emptyset$, then the following map is well-defined:

$$\Lambda : X \times N \to S^k, \qquad \lambda(x,y) := \frac{x-y}{\|x-y\|}.$$

Then $\partial(X \times N) = M \times N$ and $\lambda = \Lambda|_{\partial(X\times N)}$ is obtained as restriction of a smooth map to a boundary. Hence $\ell(M,N) = \deg(\lambda) = 0$.

c) We assume $n, m \geq 1$. Then $S^{k+1}\backslash(N \cup M)$ is nonempty and path-connected. This can be shown by an explicit construction using the tubular neighborhood theorem from Exercise 1 in charts of S^k. See also the solution of Exercise 6 where we start with a similar explicit construction. The key ingredient is that N and M both have codimension at least 2. A clean and general argument will become available, when we discuss the transversality theorem and general intersection theory. This will imply that a generic path in S^{k+1} does not intersect $N \cup M$. (This is very similar to Sard's theorem, which says that a generic point is a regular value).

Now pick a point $p \in S^{m+n+1}\backslash(N \cup M)$, denote by $\phi_p : S^{k+1}\backslash\{p\} \to \mathbb{R}^{k+1}$ the stereographic projection from the point p, and define

$$\ell_p(M,N) := \ell(\phi_p(M), \phi_p(N)).$$

More precisely, $\ell_p(M,N)$ is the degree of the map

$$\lambda_p : M \times N \to S^k, \qquad \lambda_p(x,y) := \deg\left(\frac{\phi_p(x) - \phi_p(y)}{\|\phi_p(x) - \phi_p(y)\|}\right).$$

Moving p within $S^{k+1}\backslash(N \cup M)$ to another point q, yields a homotopy from λ_p to λ_q. By homotopy invariance of the degree, it thus follows that $\ell_p(M,N)$ is independent of the choice of p.

4. Let M, N be compact manifolds. Prove that the Euler characteristic χ satisfies the following:

a) $\chi(M \times N) = \chi(M) \cdot \chi(N)$.

b) $\chi(M \cup N) = \chi(M) + \chi(N)$ when M and N are disjoint.

c) $\chi(S^{2k+1}) = 0$ and $\chi(S^{2k+2}) = 2$ for all $k \in \mathbb{N}$.

d) $\chi(T^2) = 0$.

Solution:

a) By Poincaré–Hopf theorem, $\chi(M \times N)$ does not depend on the vector field used to calculated it. We take vector fields $X_1 \in \mathrm{Vect}(M)$ and $X_2 \in \mathrm{Vect}(N)$ with isolated, non-degenerate zeroes. Denote by $p_i \in M$, $i = 1, \dots, K$ all the zeroes of X_1 and by $q_j \in N$, $j = 1, \dots, L$ all the zeroes of X_2. Recall the derivatives of the vector fields in the zeros are well-defined linear maps

$$dX_1(p_i) : T_{p_i}M \to T_{p_i}M, \qquad dX_2(p_j) : T_{q_j}N \to T_{p_j}N$$

These can be defined either using local coordinates, covariant derivatives or via embedding M and N into some euclidean space. The key observation is that they depend on none of these choices and are intrinsically well-defined. The zeros are non-degenerate if and only if these maps are isomorphism and their indices are given by the sign of the determinant.

Now take $Y = (X_1, X_2) \in \mathrm{Vect}(M \times N)$ defined by $Y(p, q) = (X_1(p), X_2(q))$ to calculate $\chi(M \times N)$. Then all the zeroes of Y are given by (p_i, q_j) for $i = 1, \dots, K$ and $j = 1, \dots, L$. Moreover,

$$dY(p_i, q_j) = dX_1(p_i) \times dX_2(q_j) : \, T_{p_i}M \times T_{q_j}N \to T_{p_i}M \times T_{q_j}N$$

is the product isomorphism. Hence Y has again non-degenerate isolated zeroes and their indices are given by

$$\begin{aligned}
\iota(Y, (p_i, q_j)) &= \mathrm{sign}(\det dY(p_i, q_j)) \\
&= \mathrm{sign}(\det d(X_1)(p_i)) \cdot \mathrm{sign}(\det d(X_2)(q_j)) \\
&= \iota(X_1, p_i) \cdot \iota(X_2, q_j)
\end{aligned}$$

for all i, j. Therefore, we calculate

$$\chi(M \times N) = \sum_{i=1}^{K} \sum_{j=1}^{L} \iota(Y, (p_i, q_j)) = \sum_{i=1}^{K} \sum_{j=1}^{L} \iota(X_1, p_i) \times \iota(X_2, q_j) = \chi(M) \cdot \chi(N).$$

b) We take any vector field Y with isolated zeroes on $M \amalg N$ (the disjoint union of M and N). Then $X_1 = Y|_M$ and $X_2 = Y|_N$ are both vector fields with isolated zeroes on M resp. N. So any zero of Y is either a zero of X_1 or of X_2 and since the index is locally defined, the indices of Y will be the same as the corresponding index for X_1 resp. X_2. So we get

$$\chi(M \amalg N) = \sum_{x \in Y^{-1}(0)} \iota(Y, x) = \sum_{x \in X_1^{-1}(0)} \iota(X_1, x) + \sum_{x \in X_2^{-1}(0)} \iota(X_2, x) = \chi(M) + \chi(N).$$

c) For any sphere $S^n \subset \mathbb{R}^{n+1}$, we can take the vector field $X(x) = (0, \dots, 0, 1) - x_{n+1}x$ for $x \in S^n$. This is a vector field with two isolated, non degenerate zeros at north and south pole. At the south pole, the differential of X is equal to the identity, so the index $\iota(X, p_S) = 1$ is equal to 1. On the other hand, the differential of X is minus the identity at the north pole and so $\iota(X, p_N) = (-1)^m$. Thus $\chi(S^m) = 1 + (-1)^m$. (This proves once more that S^2 has no vector field without a zero.)

d) We simply combine a) and c), by

$$\chi(S^1 \times S^1) = \chi(S^1) \cdot \chi(S^1) = 0.$$

5. Let $P, Q \in \mathbb{C}[x]$ be polynomials of degrees $d_P, d_Q > 0$ over $\mathbb{C}$ where P and Q have no common zeroes.

a) Define the degree for the map $P : \mathbb{C} \to \mathbb{C}$ in the usual way and check that even though $\mathbb{C}$ is not compact, the degree is independent from the regular value.

b) Define the map $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ by $f([z : 1]) = [P(z), 1]$ for $z \in \mathbb{C}$ and $f([1 : 0]) = [1 : 0]$. Prove that f is smooth and that it has degree d_P.

c) Define the map $g : \mathbb{C}P^1 \to \mathbb{C}P^1$ by $g([z : w]) = [w^d P(z/w) : w^d Q(z/w)]$ where $d = \max(d_P, d_Q)$. Prove that g is a smooth map and that it has degree d.

d) Give a map $h_k : \mathbb{C}P^1 \to \mathbb{C}P^1$ of degree k for all $k \in \mathbb{Z}$.

Solution:

a) By the fundamental theorem of algebra, each equation $P(z) = c$ has d_P solutions (counted with multiplicity.). Every regular value of P are exactly the values c such that $P(z) = c$ has only simple roots $r_1, \ldots, r_\ell$. We look at the differential $dP(r_i)\hat{z} = \Pi_{j \neq i}(r_i - r_j)\hat{z} =: \lambda_i \hat{z}$ and if we represent this as a matrix in the basis $1, i$, then we get the matrix $\begin{pmatrix} \mathrm{Re}(\lambda_i) & -\mathrm{Im}(\lambda_i) \\ \mathrm{Im}(\lambda_i) & \mathrm{Re}(\lambda_i) \end{pmatrix}$ which has determinant $|\lambda_i|^2 > 0$. Hence, it is natural to define $d_P = \deg(\mathrm{P}, \mathrm{c})$ since every root contributes positively (a general feature of holomorphic maps) and this definition is independent of the regular value c.

b) We need to see that f is smooth at $[1 : 0]$. Now assume that the non zero roots of P are $r_1, \ldots, r_\ell$ (with multiplicity) with $r_i \neq 0$, then $z^d P(1/z)$ is a polynomial of degree ℓ with roots $1/r_1, \ldots, 1/r_\ell$ and take $\epsilon < 1/|r_i|$ for $i = 1, \ldots, \ell$. So can write $\varphi_0 \circ f \circ \varphi_0^{-1}(z) = \frac{1}{P(1/z)} = \frac{z^{d_P}}{z^{d_P} P(1/z)}$ for all $z \in B_\epsilon(0) \setminus \{0\}$. This last expression is smooth on $B_\epsilon(0)$ and is zero for $z = 0$. So f is smooth. Pick a point regular value of the form $[\lambda : 1]$ which exists by Sard's theorem. We see that this means that $P(z) = \varphi_1 \circ f \circ \varphi_1^{-1}(z) = \lambda$ is a regular value of P as in a), and so the degree is equal to d_P.

c) We first check that g is well-defined even for $w = 0$.

We start by noticing that since $p(z, w) := w^d P(z/w)$ and $q(z, w) = w^d Q(z/w)$ are both homogeneous polynomials of degree d, meaning that $p(\lambda z, \lambda w) = \lambda^d p(z, w)$ and $q(\lambda z, \lambda w) = \lambda^d q(z, w)$, they define a point in $\mathbb{C}P^1$ which does not depend on the representative of the line $[z : w]$ if p and q do not vanish simultaneously. Assume $w \neq 0$, then $p(z/w, 1) = P(z/w)$ and $q(z/w, 1) = Q(z/w)$ and therefore by assumption on P and Q, these expressions cannot vanish both. Next, assume $w = 0$, then if $d_P > d_Q$, we get that $p(1, 0)$ is equal to the leading term in P which is non zero and if $d_P \leq d_Q$, then $q(1, 0) \neq 0$. So g is well defined.

For smoothness, we simply write out that

$$\varphi_1 \circ g \circ \varphi_1^{-1}(z) = \frac{P(z)}{Q(z)} \quad \text{, if } Q(z) \neq 0,$$
$$\varphi_0 \circ g \circ \varphi_1^{-1}(z) = \frac{Q(z)}{P(z)} \quad \text{, if } P(z) \neq 0,$$
$$\varphi_1 \circ g \circ \varphi_0^{-1}(w) = \frac{w^d P(1/w)}{w^d Q(1/w)} \quad \text{, if } w \neq 0 \text{ and } Q(1/w) \neq 0 \text{ or } w = 0 \text{ and } d_Q = d,$$
$$\varphi_0 \circ g \circ \varphi_0^{-1}(w) = \frac{w^d Q(1/w)}{w^d P(1/w)} \quad \text{, if } w \neq 0 \text{ and } P(1/w) \neq 0 \text{ or } w = 0 \text{ and } d_P = d.$$

All of these functions are smooth as division of polynomial functions with non zero denominator.

Finally, we need to find out the degree of g. There is $[1:\lambda] \in \mathbb{C}P^1$ for $[1:\lambda] \neq g([1:0])$ and $\lambda \neq 0$ which is a regular value of g (possible by Sard's theorem), then in a chart, we get

$$\varphi_1 \circ g \circ \varphi_1^{-1}(z) = \frac{P(z)}{Q(z)} = \lambda$$

Which means that we are looking for solutions of the polynomial equation $P(z) - \lambda Q(z) = 0$ of degree d for which 0 is a regular value. Thus by a), we get degree d.

d) For positive degrees k, we can simply take any polynomial P of degree k and apply b). For degree 0, just take a constant function. For negative degrees k, note that the function of complex conjugation $h : \mathbb{C}P^1 \to \mathbb{C}P^1$ given by $[z:w] \mapsto [\bar{z}:\bar{w}]$ has degree -1. Hence, take any map of degree $|k| > 0$ and compose it with h to get a map of degree k.

6. **a)** * Let $M^m \subset \mathbb{R}^{m+1}$ be a smooth, compact, connected manifold without boundary. Prove that M divides $\mathbb{R}^{m+1}$ into two components. The bounded component is called the interior of M and the unbounded component the exterior of M.

Hint: Define for $y \in \mathbb{R}^{m+1} \setminus M$ the map $g_y : M \to S^m : x \mapsto \frac{x-y}{|x-y|}$ and $d_y := \deg_2(g_y)$. Define two sets $V_0 := \{y \in \mathbb{R}^{m+1} \setminus M : d_y = 0\}$ and $V_1 := \{y \in \mathbb{R}^{m+1} \setminus M : d_y = 1\}$. Prove that these are the two path connected components and find out which one is unbounded.

b) Prove that a smooth, compact hypersurface $M \subset \mathbb{R}^n$ is orientable.

c) Prove that $\mathbb{R}P^2$ cannot be embedded into $\mathbb{R}^3$.

Solution:

a) We start by showing the following claim.

Claim: For each $p, q \in \mathbb{R}^{m+1} \setminus M$, we can find a path $\gamma : [0,1] \to \mathbb{R}^{m+1}$ connecting $p = \gamma(0)$ to $q = \gamma(1)$ with $|\{t \in [0,1] : \gamma(t) \in M\}| \leq 1$.

For this use exercise 1 and the tubular neighbourhood N_ϵ for some $\epsilon > 0$ small enough. Now for p, there is a shortest line segment $\ell_p : [0,1] \to \mathbb{R}^n$ to N_ϵ. Since we hit N_ϵ for the first time with this line at $p' := \ell_p(1)$, this point p' has a unique representation $p' := x_p + v_p$ for $x_p \in M$ and $v_p \in T_{x_p}M^\perp$ with $|v_p| = \epsilon$. By the same token, we get ℓ_q, q', x_q and v_q.

We see that the fibre $T_xM^\perp$ is one dimensional. Therefore, the unit normal vectors form a double cover O over M with projection $\pi : O \to M$.

As M is connected, we can lift a path connecting x_p to x_q in M to a path σ_{pq} in O with $\sigma(0) = (x_p, \frac{v_p}{|v_p|})$ and $\pi \circ \sigma_{pq}(1) = x_q$. Now we have two cases. Either $\sigma_{pq}(1) = (x_q, \frac{v_q}{|v_q|})$ or $\sigma_{pq}(1) = (x_q, -\frac{v_q}{|v_q|})$.

In the first case, we can construct a path connecting p to q by following ℓ_p, then $t \mapsto (\sigma_{pq})_x(t) + \epsilon(\sigma_{pq})_v(t)$ and finally ℓ_q. This path will not intersect M.

In the second case, we will as before follow ℓ_p, then $t \mapsto (\sigma_{pq})_x(t) + \epsilon(\sigma_{pq})_v(t)$, however now we need to cross M once to get to the 'other side'. Simply follow $t \mapsto x_q + \epsilon(2-t)(\sigma_{pq})_v(1)$ for $t \in [0,1]$. Then finally we return from $x_q + v_q$ to q by following ℓ_q. This proves our claim.

This claim already tells us that there are **at most two path connected components** of $\mathbb{R}^{m+1} \setminus M$. Start with a point $p \notin M$, then for any point $q \notin M$ the path constructed in the claim either does not cross or crosses. This defines two sets U_p (does not cross from p) and V_p (crosses once from p). U_p is clearly path connected. For V_p, we also see that for $q, r \in V_p$, we can built a connecting path by following ℓ_q, then $t \in [0,1] \mapsto (\sigma_{pq})_x(1-t) - \epsilon(\sigma_{pq})_v(1-t)$, then $t \in [0,1] \mapsto (\sigma_{pr})_x(t) - \epsilon(\sigma_{pr})_v(t)$ and then ℓ_q. (We assume we take twice the same ℓ_p. This line is by no means unique. Just think of the centre of a ball.) This does not prove that $V_p \neq \emptyset$ or that $V_p \cap U_p = \emptyset$. For that reason, we need a further topological argument.

Therefore look at the maps $g_y : M \to S^m$ for $y \in \mathbb{R}^{m+1} \setminus M$ given by $g_y(x) = \frac{x-y}{|x-y|}$ and we define $d_y := \deg_2(g_y)$. We notice that if p and q can be connected with a path γ in $\mathbb{R}^{n+1} \setminus M$, then $t \mapsto g_{\gamma(t)}$ is a homotopy between γ_p and γ_q, so $d_p = d_q$.

Now fix $y \notin M$. Let $\xi \in S^m$ be a regular value of g_y. We look at the points $y_t^\xi = y + t\xi$ for $t > 0$ which form a ray. Assume there is $x \in M$ with $g_y(x) = \xi$. Then $g_{y_t^\xi}(x) = \pm\xi$ for $y_t^\xi \notin M$ and the differential of $g_{y_t^\xi}$ at x is given by

$$dg_{y_t^\xi}(x) : T_xM \to T_{g_{y_t}(x)}S^m = \langle \xi \rangle^\perp : \hat{p} \mapsto \frac{\hat{p} \pm \langle p, \xi \rangle \xi}{|x - y_t|}.$$

So $dg_{y_t^\xi}(x)$ is surjective (the same as injective) if and only if $dg_y(x)$ is surjective if and only if $\xi \notin T_xM$. Hence we get the criterion that ξ is a regular value for g_y if and only if ξ is a regular value of $g_{y_t^\xi}$ for y_t^ξ if and only if $\xi \notin T_xM$ for all $x \in g_y^{-1}(\xi)$.

We want to see now what happens to d_y when we 'cross' M. Pick $y \notin M$ and choose ξ such that there is $T > 0$ $p + T\xi \in M$. By Sard's theorem, we can assume by wiggling ξ a bit, that this is a regular value. Denote by $x_i = y + t_i\xi \in M$ the intersection points of M with the ray with $t_1 < t_2 < \ldots < N$. Then $g_{y_t^\xi}(x_1) = \xi$ for all $t < t_1$ and $g_{y_t^\xi}(x_1) = -\xi$ for all $t_1 < t < t_2$ whereas $g_{y_t^\xi}(x_i) = \xi$ for all $i = 2, \ldots, N$ for all $t < t_2$. Hence $|d_y - d_{y_{t'}^\xi}| = 1$ for $t_1 < t' < t_2$.

In conclusion, $V_0 := \{y \in \mathbb{R}^{m+1} \setminus M : d_y = 0\}$ and $V_1 := \{y \in \mathbb{R}^{m+1} \setminus M : d_y = 1\}$ are non empty, open, disjoint and $M = V_0 \cup V_1$. These have to be two connected components by the argument at the start. Since M is compact, it is bounded i.e. $M \subset B_R(0)$. One can see that if $|y| \geq R$, then g_y is not surjective. Indeed, take any $\xi \perp y$ and see that the ray $p + t\xi \notin B_R(0)$ for all $t > 0$ and so has no intersection with M. By the previous discussion, this means that ξ is regular and $d_y = 0$. Hence

V_1 is bounded.

b) We need to show that there is a non-zero unit normal vector ν to M. As discussed in a), the unit normal bundle $\pi : O \to M$ is a two sheeted cover. Now take $\epsilon > 0$ small enough, such that N_ϵ is a tubular neighbourhood as in a). Then we can define $\nu(p) \in O_p$ for $p \in M$ by saying that $p + \epsilon\nu(p) \in V_0$. This is well-defined and smooth. We can now orient every tangent space T_pM by saying that the base $(e_1, \ldots, e_m)$ is positive if $(\nu(p), e_1, \ldots, e_m)$ is positive in $\mathbb{R}^{m+1}$.

c) By b), we know that if $\mathbb{R}P^2 \subset \mathbb{R}^3$, then $\mathbb{R}P^2$ would be orientable. So we simply need to prove that $\mathbb{R}P^2$ is non orientable. The main point here is that the antipodal map on S^2 is orientation reversing.

Thus if you look at $\mathbb{R}P^2$ as the quotient of S^2 by the antipodal map $\pi(x) = -x$, then we can look at the path $\pi \circ \gamma$ where γ is given by $t \mapsto (\cos(\pi t), \sin(\pi t), 0)$ for $t \in [0, 1]$. This is a loop and we can consider for $0 \le t < 1$ that the images under $d\pi$ of $e_1 = \dot\gamma$ and the vector 'pointing upwards' $e_2 = (0, 0, 1)$ form a positive basis. The interesting thing now happens if we try to close up the loop. Namely, we use π to identify the point $(1, 0, 0)$ and $(-1, 0, 0)$. We then use the differential of π to identify the tangent vectors at $(-1, 0, 0)$ to tangent vectors at $(1, 0, 0)$. Under the antipodal map, we get that $d\pi(-1, 0, 0)e_1(1) = e_1(0)$, but $d\pi(-1, 0, 0)e_2(1) = -e_2(0)$. This means that there is no coherent way of choosing an orientation along this loop in $\mathbb{R}P^2$. If we take a tubular neighbourhood of the loop $\pi \circ \gamma$, then we identified a Möbius band in $\mathbb{R}P^2$ which proves that $\mathbb{R}P^2$ is a non-orinietable manifold.