Solution 5

1. **a)** Find a surjective map $f : S^1 \to S^1$ with $\deg(f) = 0$.

b) For every $k \in \mathbb{Z}$ and $m \geq 1$ find a map $g_k : S^m \to S^m$ with $\deg(g_k) = k$.

Solution:

a) Take the map $f : S^1 \to S^1$ to be the composition of $z \mapsto \frac{1-3/4z}{|1-3/4z|}$ (degree 0; the image contains the arc $[-\pi/4, \pi/4]$) composed with the map $z \mapsto z^6$ (degree 6; surjective even if restricted to $[-\pi/4, \pi/4]$). f has degree $0 = 0 \cdot 6$.

b) Let $h_{k,1} : S^1 \to S^1$ be the map $h_{k,1}(z) = z^k$. We have seen in Exercise 1 of Exercise Sheet 3 that $\deg(h_{k,1}) = k$. We define iteratively function $h_{k,m} : S^m \to S^m$ for every $m \geq 1$ with $\deg(h_{k,m}) = m$. Define the projection of the double punctured sphere to the cylinder by

$$\Phi_m : S^m \setminus \{p_S, p_N\} \to S^{m-1} \times (-1, 1)$$
$$\Phi_m(x_1, \ldots, x_m, x_{m+1}) := \left(\frac{(x_1, \ldots, x_m)}{\sqrt{x_1^2 + \cdots + x_m^2}}, x_{m+1} \right).$$

This is an orientation preserving diffeomorphism. Assume we already have defined $h_{m-1,k} : S^{m-1} \to S^{m-1}$ of degree k, and define $h_{m,k} : S^m \to S^m$ by

$$h_{m,k}(x) := \Phi_m^{-1} \left(h_{m-1,k} \left(\frac{(x_1, \ldots, x_m)}{\sqrt{x_1^2 + \cdots + x_m^2}} \right), x_{m+1} \right) \qquad \text{for } x \in S^m \setminus \{p_S, p_N\},$$
$$h_{m,k}(p_S) = p_S, \qquad h_{m,k}(p_N) = p_N.$$

This map is just a k-fold rotation of S^m around the axis through the north and south pole and it is smooth even at the poles. To calculate the degree, let y be a regular value of $h_{m-1,k}$, then $(y, 0)$ is a regular value of $h_{m,k}$ as we have $\mathrm{id}'(0) = 1$. Furthermore, there is the same number of preimages and the signs of the differentials match as Φ_m is orientation preserving. So $\deg(h_{m,k}) = k$.

2. Let M be a compact, connected, non-orientable manifold without boundary.

a) Let $p, q \in M$ be two points and choose bases $X = (X_1, \ldots, X_m) \subset T_pM$ and $X' = (X'_1, \ldots, X'_m) \subset T_qM$. Show that (p, X) and (q, X') are framed cobordant.

b) Show that two framed 0-dimensional submanifolds $(N = \{p_1, \ldots, p_\ell\}, X)$ and $(N' = \{q_1, \ldots, q_k\}, X')$ are framed cobordant if and only if the number of points $\#N$ and $\#N'$ agrees modulo two.

c) Show that two maps $f, g : M \to S^m$ are homotopic if and only if they have the same mod 2 degree.

Hint: Part a) uses similar ideas as the solution of Exercise 6 b) of Exercise Sheet 3.

Solution:

a) We claim that there exists a path $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$ and a moving frame $X(t) = (X_1(t), \ldots, X_m(t)) \subset T_{\gamma(t)}M$ along γ such that $X(0) = X$ and $X(1)$ induces the same orientation as X' on T_qM.

We show first how part a) follows from this claim: After reparametrization, we may assume that $\gamma(t) = p$ for $0 \le t \le \epsilon$, $\gamma(t) = q$ for $1 - \epsilon \le t \le 1$. Then

$$C := \{(t, \gamma(t)) \in [0,1] \times M \,|\, 0 \le t \le 1\}, \qquad \partial C = (\{0\} \times \{p\}) \cup (\{1\} \times \{q\})$$

is a cobordism between p and q. Since $X(1)$ and X' induce the same orientation on T_qM, they are related by $X(1)g = X'$ for some $g \in \mathrm{GL}^+(m, \mathbb{R})$. Since $\mathrm{GL}^+(m, \mathbb{R})$ is path-connected, there exists a smooth path $[0,1] \to \mathrm{GL}^+(m, \mathbb{R})$, $t \mapsto g(t)$, such that $g(0) = \mathbb{1}$ and $g(1) = g$. After reparametrization we may assume that $g(0) = \mathbb{1}$ for $0 \le t \le \epsilon$ and $g(1) = g$ for $1 - \epsilon \le t \le 1$. Then $X''(t) := X(t)g(t)$ is a framing along C, always transverse to the tangent spaces TC, and agrees with X and X' near the boundaries. Thus (C, X'') is a framed cobordism between (p, X) and (q, X').

We prove the claim next. Given any path $\gamma : [0,1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$, there exists a moving frame $X(t) = (X_1(t), \ldots, X_m(t)) \subset T_{\gamma(t)}M$ along γ with $X(0) = X$. This can be defined by first choosing a Riemannian metric, or an embedding $M \subset \mathbb{R}^n$, and then using parallel transport of the Levi-Civita connection. We saw in Step 1 of the solution of Exercise 6 b) on Exercise Sheet 3, that the orientation of $X(1)$ depends only on γ and the initial frame X. Now suppose by contradiction that all paths $\gamma : [0,1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$ yield the same orientation in T_qM. Then Step 2 of the solution of Exercise 6 b) on Exercise Sheet 3 is trivially satisfied and it follows from Step 3 that M is orientable. This contradicts our assumption that M is non-orientable. Hence for each orientation of T_qM there exists a path γ and a moving frame $X(t)$ along $\gamma(t)$ such that $X(1)$ represents the given orientation. This proves the claim.

b) Let $(\{p_1, \ldots, p_\ell\}, X)$ be a framed 0-dimensional submanifold of M and assume $\ell \ge 2$. We show first that $(\{p_1, \ldots, p_\ell\}, X)$ is framed cobordant to $(\{p_1, \ldots, p_{\ell-2}\}, X)$. Let $\gamma : [0,1] \to M$ be a smooth path with $\gamma(t) = p_{\ell-1}$ for $0 \le t \le \epsilon$, $\gamma(t) = p_\ell$ for $1 - \epsilon \le t \le 1$. We may assume that $\gamma(t) \in M \backslash \{p_1, \ldots, p_{\ell-2}\}$ for all t. Then

$$C := \{(t(1-t), \gamma(t)) \,:\, 0 \le t \le 1\} \cup ([0,1] \times \{p_1, \ldots, p_{\ell-2}\}) \subset M \times [0,1]$$

is a smooth manifold with boundary

$$\partial C = (\{0\} \times \{p_1, \ldots, p_\ell\}) \cup (\{1\} \times \{p_1, \ldots, p_{\ell-2}\}).$$

Hence $\{p_1, \ldots, p_\ell\}$ and $\{p_1, \ldots, p_{\ell-2}\}$ are cobordant. A framing for this cobordism is obtained by taking constant frames $X(p_i)$ at (t, p_i) for $i = 1, \ldots, \ell - 2$ along the constant pieces. For the remaining component we use parallel transport in $[0,1] \times M$ to move the initial frame

$$((0, X_1(p_{\ell-1})), \ldots, (0, X_m(p_{\ell-1}))) \subset (T_{(0,p_{\ell-1})}C)^\perp \subset \mathbb{R} \times T_{p_{\ell-1}}M = T_{(0,p_{\ell-1})}([0,1] \times M)$$

along the arc $\alpha(t) := (t(1-t), \gamma(t))$ to obtain a framing

$$((\tau(t), X_1(t)), \ldots, (\tau(t), X_m(t))) \subset (T_{\alpha(t)}C)^\perp.$$

For $t = 1$ this has the shape $((0, X_1(1)), \ldots, (0, X_m(1)))$ and so corresponds to a basis of $T_{p_\ell}M$. A similar argument as in part a) shows that the orientation of this basis depends only on the choice of γ, and there exists a choice for which this basis represents the same orientation as $X(p_\ell)$. This shows that $(\{p_1,\dots,p_\ell\},X)$ is framed cobordant to $(\{p_1,\dots,p_{\ell-2}\},X)$.

It follow by induction on ℓ that a framed 0-dimensional submanifold $(\{p_1,\dots,p_\ell\},X)$ is framed cobordant to (p_1,X) when ℓ is odd and framed null-cobordant (i.e. framed cobordant to the emptyset) when ℓ is even. Since (p_1,X) and (q_1,X') are framed cobordant by part (a), it follows that N and N' are framed cobordant when $\#N$ and $\#N'$ are both odd. Conversely, when $\#N$ and $\#N'$ are both even, they are both framed cobordant to the empyset and by transitivity they are also cobordant to each other.

Finally, we show that (p_1,X) is not null-cobordant. A cobordism between (p_1,X) and the emptyset would be a compact 1-dimensional manifold $C\subset[0,1]\times M$ with $\partial C=\{0\}\times\{p_1\}$. But we know from the classification of compact 1-manifolds, that for every compact 1-manifold $\#\partial C$ is even. Hence such a cobordism does not exists and (p_1,X) is not null-cobordant. (Note that this argument does not use the framing)

c) The main result for the Pontryagin construction (Theorem C in Milnor's book) says that f,g are homotopic if and only if $(f^{-1}(y),X)$ and $(g^{-1}(z),X')$ are framed cobordant for any choice of regular values. The framings X and X' are naturally induced from the orientation of S^m. It follows from part b) that $(f^{-1}(y),X)$ and $(g^{-1}(z),X')$ are framed cobordant if and only if $\deg_2(f)=\deg_2(g)$. Hence f,g are homotopic if and only if $\deg_2(f)=\deg_2(g)$.

3. Let M be a compact, connected, oriented manifold without boundary.

a) Let $p,q\in M$ be two points and choose a bases $X=(X_1,\dots,X_m)\subset T_pM$ and $X'=(X_1',\dots,X_m')\subset T_qM$. Show that (p,X) and (q,X') are framed cobordant if and only if X and X' are both positive or both negative bases.

b) Show that two framed 0-dimensional submanifolds $N=(\{p_1,\dots,p_\ell\},X)$ and $N'=(\{q_1,\dots,q_k\},X')$ are framed cobordant if and only if the number of points counted with signs

$$\sum_{i=1}^{\ell}\operatorname{sign}(X,p_i)=\sum_{j=1}^{k}\operatorname{sign}(X',q_j)$$

agrees for both framed submanifolds.

c) Deduce Hopf's theorem: Two maps $f,g:M\to S^m$ are homotopic if and only if they have same degree.

Hint: Part a) uses similar ideas as the solution of Exercise 6 b) of Exercise Sheet 3.

Solution:

a) Choose a path $\gamma:[0,1]\to M$ such that $\gamma(t)=p$ for $0\le t\le\epsilon$ and $\gamma(t)=q$ for $q-\epsilon\le t\le 1$. Then there exists a moving frame $X(t)=(X_1(t),\dots,X_m(t))\subset T_{\gamma(t)}M$ with $X(t)=X(p)$ for $0\le t\le\epsilon$. This can be defined by first choosing a Riemannian metric, or an embedding $M\subset\mathbb{R}^n$, and then using parallel transport of the Levi-Civita connection.

We claim that $X(t)$ positive basis of $T_{\gamma(t)}M$ if and only if $X(0)$ is a positive basis T_pM. For this let $t_0\in[0,1]$ and choose a chart $\phi:U\to\mathbb{R}^m$ defined on a neighborhood of $T_{\gamma(t_0)}$ which is compatible with the orientation of M. This defines a smooth map

$$I_{t_0}\to\mathrm{GL}(m,\mathbb{R}),\qquad t\mapsto d\phi(\gamma(t))X(t)$$

defined on some open interval $t_0 \in I_{t_0} \subset \mathbb{R}$, where we identify a basis in $\mathbb{R}^n$ with an invertible $n \times n$ matrix. It follows for $t \in I_{t_0}$, that $X(t)$ is a positive basis if and only if $\det(d\phi(\gamma(t))X(t)) > 0$. By continuity this is the case if and only if $\det(d\phi(\gamma(t_0))X(t_0)) > 0$. This shows that

$$\{t \in [0,1] \,|\, X(t) \text{ is positive}\}, \qquad \{t \in [0,1] \,|\, X(t) \text{ is negative}\}$$

are both open and closed subsets. Since $[0,1]$ is connected, it follows that one of these sets is empty and in particular $X(1)$ is positive resp. negative if and only if $X(0) = X(p)$ is positive resp. negative.

We construct a framed cobordism between (p, X) and (q, X') provided that the orientation of p and q are both positive or negative. Let γ be as above and define

$$C := \{(t, \gamma(t)) \in [0,1] \times M \,:\, 0 \le t \le 1\}, \qquad \partial C = (\{0\} \times \{p\}) \cup (\{1\} \times \{q\})$$

This is a cobordism between p and q. Let $X(t)$ be a moving frame along $\gamma(t)$ with $X(0) = X(p)$. We have seen above that $X(1)$ and $X'(q)$ induce the same orientation on T_qM, and hence they are related by $X(1)g = X'(q)$ for some $g \in \mathrm{GL}^+(m, \mathbb{R})$. Since $\mathrm{GL}^+(m, \mathbb{R})$ is path-connected, there exists a smooth path $[0,1] \to \mathrm{GL}^+(m, R)$, $t \mapsto g(t)$, such that $g(0) = \mathbb{1}$ and $g(1) = g$. After reparametrization we may assume that $g(0) = \mathbb{1}$ for $0 \le t \le \epsilon$ and $g(1) = g$ for $1 - \epsilon \le t \le 1$. Then $X''(t) := X(t)g(t)$ is a framing along C which agrees with X and X' near the boundaries and thus (C, X'') is a framed cobordism between (p, X) and (q, X').

Finally, suppose that X is a positive basis of T_pM and X' is a negative basis of T_qM. We need to show that (p, X) is not framed cobordant to (q, X'). Suppose that

$$C \subset [0,1] \times M, \qquad \partial C = (\{0\} \times \{p\}) \cup (\{1\} \times \{p\})$$

is a cobordism between p and q. A framing X for C assigns to every $(t, x) \in C$ a bases $X(t, x) = (X_1(t, x), \ldots, X_m(t, x)) \subset T_{(t,x)}C^\perp \subset \mathbb{R} \times T_xM$. Let $C_0 \subset C$ be the connected component which contains $\{0\} \times \{p\}$. Then, by the classification of compact 1-manifolds, C_0 is an arc and there exists a parametrization

$$C_0 = \{(\rho(t), \gamma_0(t)) \,:\, 0 \le t \le 1\}.$$

The second boundary point of C_0 must be $\{1\} \times \{q\}$, since C has only two boundary points. It follows that $\rho(0) = 0$, $\rho(1) = 1$ and γ_0 is a smooth curve with $\gamma_0(0) = p$ and $\gamma_0(1) = q$. After reparametrization, we may assume that ρ is strictly monotone increasing. Then the projection of $T_{(\rho(t), \gamma_0(t))}C^\perp$ onto $T_{\gamma_0(t)}M$ is bijective and hence each frame $X(\rho(t), \gamma_0(t))$ determines a frame $X''(t)$ of $T_{\gamma_0(t)}M$. This yields a moving frame along $\gamma_0(t)$ where $X''(0) = X(p)$ is positive and $X''(1) = X(1)$ is negative. This is impossible by our discussion above, and hence (p, X) and (q, X') are not framed cobordant.

b) Let $(\{p_1, \ldots, p_\ell\}, X)$ be a framed 0-dimensional submanifold of M and assume that $X(p_\ell)$ is a negative of $T_{p_\ell}M$ and $X(p_{\ell-1})$ a positive basis of $T_{p_{\ell-1}}M$. We show that $(\{p_1, \ldots, p_\ell\}, X)$ is then framed cobordant to $(\{p_1, \ldots, p_{\ell-2}\}, X)$. Let $\gamma : [0,1] \to M$

be a smooth path with $\gamma(t) = p_{\ell-1}$ for $0 \le t \le \epsilon$, $\gamma(t) = p_\ell$ for $1-\epsilon \le t \le 1$. We may assume that $\gamma(t) \in M \backslash \{p_1, \dots, p_{\ell-2}\}$ for all t. Then

$$C := \{(t(1-t), \gamma(t)) \,:\, 0 \le t \le 1\} \cup ([0,1] \times \{p_1, \dots, p_{\ell-2}\}) \subset M \times [0,1]$$

is a smooth manifold with boundary

$$\partial C = (\{0\} \times \{p_1, \dots, p_\ell\}) \cup (\{1\} \times \{p_1, \dots, p_{\ell-2}\}).$$

Hence $\{p_1, \dots, p_\ell\}$ and $\{p_1, \dots, p_{\ell-2}\}$ are cobordant. A framing for this cobordism is obtained by taking the constant frames $X(p_i)$ along the constant pieces $[0,1] \times \{p_i\}$ for $i = 1, \dots, \ell - 2$. For the remaining component we use parallel transport in $[0,1] \times M$ to move the initial frame

$$((0, X_1(p_{\ell-1})), \dots, (0, X_m(p_{\ell-1}))) \subset (T_{(0,p_{\ell-1})}C)^\perp \subset \mathbb{R} \times T_{p_{\ell-1}}M = T_{(0,p_{\ell-1})}([0,1] \times M)$$

along the arc $\alpha(t) := (t(1-t), \gamma(t))$ to obtain a framing

$$((\tau(t), X_1(t)), \dots, (\tau(t), X_m(t))) \subset (T_{\alpha(t)}C)^\perp.$$

Then $(\dot\alpha(t), (\tau(t), X_1(t)), \dots, (\tau(t), X_m(t)))$ is a basis of $T_{\alpha(t)}([0,1] \times M)$. This basis is positive for $t = 0$ and hence for all t, by our discussion in part a) now applied to $[0,1] \times M$ instead of M. In particular, $((-1,0), (0, X_1(1)), \dots, (0, X_m(1)))$ is a positive basis of $T_{(1,p_\ell)}([0,1] \times M)$ and hence $(X_1(1), \dots X_m(1))$ is a negative basis of $T_{p_\ell}M$. Hence we can modify our framing to agree with $X(p_\ell)$ and this proves the claim.

It follows that any framed 0-dimensional submanifolds $(\{p_1, \dots, p_\ell\}, X)$ is framed cobordant to a submanifold where all points p_i have the same sign, i.e. all bases $X(p_i)$ have the same sign. Now suppose that $N = (\{p_1, \dots, p_\ell\}, X)$ and $N' = (\{q_1, \dots, q_\ell\}, X')$ are two framed 0-dimensional submanifolds of the same cardinality, with all points having the same sign. By part (a) there exists for each i an cobordism

$$C_i \subset [0,1] \times M, \qquad \partial C_i = (\{0\} \times \{p_i\}) \cup (\{1\} \times \{q_i\})$$

with framing X_i matching $X(p_i)$ and $X'(q_i)$ near the boundary. Looking at the custruction in a), we see that there is a lot of freedom in the construction, and it is not hard to construct these cobordism in a way such that $C_i \cap C_j = \emptyset$. (Either make all the curves γ_i disjoint, or reparametrize the curves such that $\gamma_i(t) \ne \gamma_j(t)$ for all t) Then the disjoint union

$$C = \bigcup_{i=1}^{\ell} C_i, \qquad \partial C = (\{0\} \times \{p_1, \dots, p_\ell\}) \cup (\{1\} \times \{q_1, \dots, q_\ell\})$$

is a cobordism between N and N' and the framing of each C_i defines a framing for C compatible with the framings of N and N'.

Conversely, suppose that $N = (\{p_1, \dots, p_\ell\}, X)$ and $N' = (\{q_1, \dots, q_k\}, X')$ are framed cobordant. By our discussion above, we may assume that all points of N and N' have respectively the same sign. Denote the cobordism by

$$C \subset [0,1] \times M, \qquad \partial C = (\{0\} \times \{p_1, \dots, p_\ell\}) \cup (\{1\} \times \{q_1, \dots, q_k\}).$$

Let $C_i \subset C$ be the connected component which contains $(0, p_i)$. By the classification of 1-manifolds C_i is an arc. We claim that the second endpoint of this arc cannot be any of the points $(0, p_j)$. Indeed, it would follow from our discussion above, that the frame $X(p_j)$ would have the opposite sign as the frame of $X(p_i)$. But we assumed that all frames for N have the same sign. Hence the second boundary point is one of the points $(0, q_j)$. In particular, C_i is a framed cobordism between (p_i, X) and (q_j, X'). It follows from part (a) that $X'(q_j)$ and $X(p_i)$ have the same sign. Moreover, the cobordism C connects every point of N to precisely one point of N' Hence, we see that $\#N \leq \#N'$. Reversing the roles of N and N', it follows that $\#N = \#N'$ and all points have the same sign.

c) The main result for the Pontryagin construction (Theorem C in Milnor's book) says that f, g are homotopic if and only if $(f^{-1}(y), X)$ and $(g^{-1}(z), X')$ are framed cobordant for any choice of regular values. The framings X and X' are naturally induced from the orientation of S^m. It follows from part b) that $(f^{-1}(y), X)$ and $(g^{-1}(z), X')$ are framed cobordant if and only if $\deg(f) = \deg(g)$. Hence f, g are homotopic if and only if $\deg(f) = \deg(g)$.

4. Let M and N be the following two circles in $\mathbb{R}^3$

$$M = \{x^2 + y^2 = 1,\ z = 0\}, \qquad N = \{(x-1)^2 + z^2 = 1,\ y = 0\}.$$

Fix orientations of M and N and compute the linking number $\ell(M, N)$ for these orientations. (See Exercise 4, Serie 4)

Solution: Here is a picture of our situation.

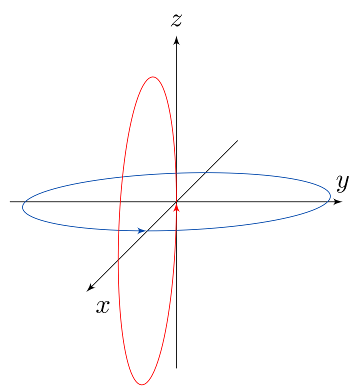

Figure 1: The two linking disks. M is blue and N is red.

We choose embeddings of S^1 given by

$$f : S^1 \to M : e^{i\theta} \mapsto (\cos(\theta), \sin(\theta), 0)$$

and

$$g : S^1 \to N : e^{i\varphi} \mapsto (1 - \cos(\varphi), 0, \sin(\varphi)).$$

These also fix the orientations, as indicated by the arrows on the figure. The linking number $\ell(M, N)$ is defined as the degree of the linking map $\lambda : S^1 \times S^1 \to S^2$ defined by

$$\lambda(e^{i\theta}, e^{i\varphi}) := \frac{1}{\sqrt{3 - 2\cos(\theta) - 2\cos(\varphi) + 2\cos(\theta)\cos(\varphi)}} \begin{pmatrix} \cos(\theta) - 1 + \cos(\varphi) \\ \sin(\theta) \\ -\sin(\varphi) \end{pmatrix}.$$

We see that $\lambda^{-1}(0,0,-1) = \{(e^0, e^{\frac{\pi}{2}i}) = (1,i)\}$. A little calculation gives

$$d\lambda(1,i)(\hat{\theta},\hat{\varphi}) = (-\hat{\varphi},\hat{\theta},0)$$

which is surjective, therefore $(0,0,-1)$ is a regular value of λ. To determine the sign, we need to know the sign of the basis $(e_2,-e_1)$ of $T_{p_S}S^2$. For this we calculate

$$\det\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = -1.$$

Thus $\ell(M,N) = \deg(\lambda) = -1$.

5. The Hopf invariant. If $y \neq z$ are regular values for a map $f : S^{2n-1} \to S^n$ for $n \geq 2$, then the manifolds $f^{-1}(y)$ and $f^{-1}(z)$ can be oriented. Hence the linking linking number $\ell(f^{-1}(y), f^{-1}(z))$ is defined. (See exercise 4 c) of sheet 4).

a) Explain how an orientation for $f^{-1}(y)$ and $f^{-1}(z)$ is obtained.

b) Prove that the linking number is locally constant as a function of y.

c) If y and z are regular values of $g : S^{2n-1} \to S^n$ also, where

$$\|f(x) - g(x)\| < \|y - z\|$$

for all x, prove that

$$\ell(f^{-1}(y), f^{-1}(z)) = \ell(g^{-1}(y), f^{-1}(z)) = \ell(g^{-1}(y), g^{-1}(z)).$$

d) Prove that the linking number $\ell(f^{-1}(y), f^{-1}(z))$ depends only on the homotopy class of f, and does not depend on the choice of y and z.

This integer $H(f) := \ell(f^{-1}(y), f^{-1}(z))$ is called the **Hopf invariant** of f.

Solution:

a) Note that the spheres S^{2n-1} and S^n have a natural orientation and $\dim f^{-1}(y) = n-1$. Let $\nu = (\nu_1,\dots,\nu_n)$ be a positive basis of T_yS^n. For every $p \in f^{-1}(y)$ the differential $df(p) : T_pS^{2n-1} \to T_yS^n$ restricts to an isomorphism from $(T_pf^{-1}(y))^\perp$ onto T_yS^n. Hence there exists a unique basis $(f^*\nu)(p) = (f^*\nu_1,\dots,f^*\nu_n)$ of $(T_pf^{-1}(y))^\perp$ which gets maped to ν under $df(p)$. This is the natural framing of $f^{-1}(y) \subset M$. Now we say that a basis $(v_1,\dots,v_{n-1})$ of $T_pf^{-1}(y)$ is positive if the combined bases $(v_1,\dots,v_{n-1},f^*\nu_1,\dots,f^*\nu_n)$ is a positive basis of T_pS^{2n-1}. This defines an orientation on $f^{-1}(y)$.

b) We start by stating a claim, which we prove later.

Claim: If M, M', N are all $(n-1)-$dimensional framed submanifolds of S^{2n-1} with M and M' being framed cobordant via $X \subset S^{2n-1} \times [0,1]$ with $X \cap (N \times [0,1]) = \emptyset$, then $\ell(M,N) = \ell(M',N)$.

We now show how this yields the result. The set of critical points for any smooth function $f : S^{2n-1} \to S^n$ C is closed, as surjectivity is an open condition. Since S^{2n-1} is compact, f sends closed set to closed sets and thus the set of regular values $S^{2n-1} \setminus f(C)$ is open. It follows for sufficiently small $\epsilon > 0$ that $B_\epsilon(y) \subset S^{2n-1} \setminus (f(C) \cup \{z\})$ (Here one can use either the extrinsic or intrinsic distance on S^{2n-1} to define the ball ϵ-ball around y). For $y' \in B_\epsilon(y)$,

$$M = f^{-1}(y), \qquad M' = f^{-1}(y'), \qquad N = f^{-1}(z)$$

$$X = \left\{(p,t) \in S^{2n-1} \times [0,1] \;:\; f(p) = \frac{ty + (1-t)y'}{|ty + (1-t)y'|}\right\}$$

fulfill the assumptions of the claim. Hence, $\ell(M', N) = \ell(M, N)$.

So let us prove this claim. Since $n \geq 2$, we have $\dim(X) = n < \dim(S^{2n-1})$. It follows from Sard's theorem that the projection $X \to S^{2n-1}$, $(t,p) \mapsto p$, is not surjective and, by compactness of X, its image is closed. Hence there exists a point $p \in S^{2n-1} \backslash N$ such that $(\{p\} \times [0,1]) \cap X = \emptyset$. Let $\pi : S^{2n-1} \to \mathbb{R}^{2n-1}$ be the stereographic projection with respect to p and define

$$\mathcal{M} := \pi(M),\ \mathcal{M}' := \pi(M'),\ \mathcal{N} := \pi(N)\ \subset \mathbb{R}^{2n-1},$$

$$\mathcal{X} := (\pi \times \mathrm{id})(X) \subset \mathbb{R}^{2n-1} \times [0,1]$$

Now consider the map

$$\Lambda : \mathcal{X} \times \mathcal{N} \to S^{2n-2} : ((x,t), y) \mapsto \frac{x-y}{|x-y|}.$$

Note that

$$\partial(\mathcal{X} \times \mathcal{N}) = (\mathcal{M} \times \{0\} \times \mathcal{N}) \cup (\mathcal{M}' \times \{1\} \times \mathcal{N})$$

and Λ extends the map $\lambda_0 : \mathcal{M} \times \mathcal{N} \to S^{2n-2}$, the map whose degree defines $\ell(\mathcal{M}, \mathcal{N}) = \ell(M, N)$, and the map $\lambda_1 : \mathcal{M}' \times \mathcal{N} \to S^{2n-2}$, the map whose degree defines $\ell(\mathcal{M}', \mathcal{N}) = \ell(M', N)$, on its boundary. The framing of $\mathcal{X}$ and the natural orientation of $\mathbb{R}^{2n-1} \times [0,1]$ induce a orientation of $\mathcal{X}$ as in part (a). The induced boundary orientation on $\partial\mathcal{X}$ yields on $\mathcal{M}$ the same orientation as in part (a) and on $\mathcal{M}'$ the opposite orientation. Hence

$$0 = \deg(\Lambda|_{\partial(\mathcal{X}\times\mathcal{N})}) = \deg(\lambda_0) - \deg(\lambda_1) = \ell(M,N) - \ell(M',N)$$

and this proves the claim.

c) We see that the second equality follows from the first by using the symmetry in exercise 4 a) of sheet 4 and exchanging the role of (g, y) and (f, z). So we only need to prove the equality $\ell(f^{-1}(y), f^{-1}(z)) = \ell(g^{-1}(y), f^{-1}(z))$.

Note that in particular

$$\|f(x) - g(x)\| < \|y - z\| \leq 2$$

for all $x \in S^{2n-1}$. Hence we know that there is the homotopy

$$H(p,t) = \frac{tf(p) + (1-t)g(p)}{|tf(p) + (1-t)g(p)|}$$

between f and g. Take a common regular value y' of H, f and g close to y such that the condition $|f(x) - g(x)| < |y' - z|$ for all $x \in S^{2n-1}$ still holds. Due to this latter condition, $H^{-1}(y') \cap f^{-1}(z) = \emptyset$. Thus we can apply the claim of b) with

$$M = f^{-1}(y'),\ M' = g^{-1}(y'),\ X = H^{-1}(y'),\ \text{and}\ N = f^{-1}(z).$$

Therefore, $\ell(g^{-1}(y'), f^{-1}(z)) = \ell(f^{-1}(y'), f^{-1}(z))$. By b) both sides of the equation are locally constant as functions of y', and thus it follows $\ell(g^{-1}(y), f^{-1}(z)) = \ell(f^{-1}(y), f^{-1}(z))$.

d) Let H be a homotopy from f to g. Since $S^n \times [0,1]$ is compact, $|H(p,t) - H(p,t')| < 1/2|y-z|$ whenever $|t-t'| < \delta$. Therefore, if we take $0 = t_0 < t_1 < \ldots < t_N = 1$ such that $|t_i - t_{i-1}| < \delta$. Choose y' close to y and z' close to z such that z', y' are regular values of $h_i : S^{2n-1} \to S^n : p \mapsto H(p,t_i)$ for $i = 0, \ldots, N$ and H.

Here close means $|z'-z| < \frac{1}{4}|y-z|$, $|y'-y| < \frac{1}{4}|y-z|$, $\ell(f^{-1}(y), f^{-1}(z)) = \ell(f^{-1}(y'), f^{-1}(z'))$ and $\ell(g^{-1}(y), g^{-1}(z)) = \ell(g^{-1}(y'), g^{-1}(z'))$. Then $\frac{1}{2}|y-z| \leq |y'-z'|$ and we can apply c) to get $\ell(h_i^{-1}(y'), h_i^{-1}(z')) = \ell(h_{i-1}^{-1}(y'), h_{i-1}^{-1}(z'))$ for $i = 1, \ldots, N$. In particular, $\ell(g^{-1}(y'), g^{-1}(z')) = \ell(f^{-1}(y'), f^{-1}(z'))$.

To prove that $\ell(f^{-1}(y), f^{-1}(z))$ is independent of the choice of y and z, the idea is to construct an isotopy $\Psi : S^n \times [0,1] \to S^n$ such that $\Psi_0 = \mathrm{id}$, Ψ_t is a diffeomorphism for $t \in [0,1]$, $\Psi_1(y) = y'$ and $\Psi_1(z) = z$ for $y' \neq z$ another regular value of f. With this we can define a homotopy $H : S^{2n-1} \times [0,1] \to S^n : (p,t) \mapsto h_t(p) := \Psi_t(f(p))$ and conclude from the homotopy invariance shown above that

$$\ell(f^{-1}(y'), f^{-1}(z)) = \ell(h_0^{-1}(y'), h_0^{-1}(z)) = \ell(h_1^{-1}(y'), h_1^{-1}(z)) = \ell(f^{-1}(y), f^{-1}(z)).$$

We construct the isotopy next. We may assume that y and y' are not antipodal. Then the isotopy can be constructed by cutting off the vector field $X \in \mathrm{Vect}(S^{2n-1})$ with $X_{(y,y')}(p) := (y'-y) - \langle y'-y, p\rangle p$. I.e. take a smooth function $\rho : S^n \to \mathbb{R}$ such that $\rho(p) > 0$ for all $p \neq z$ and $\rho(z) = 0$. Then we take Ψ_t to be the time t map of the flow of $Y := \rho X_{(y,y')}$. Then y, y' lie on the same integral curve by construction and up to changing t to λt for $\lambda > 0$, we get that $\Psi_1(y) = y'$ and $\Psi_1(z) = z$.

6. **a)** If the dimension n is odd, prove that $H(f) = 0$.

b) For a composition

$$S^{2n-1} \xrightarrow{f} S^n \xrightarrow{g} S^n$$

prove that $H(g \circ f) = deg(g)^2 \cdot H(f)$.

c) The **Hopf fibration** $\pi : S^3 \to S^2$ is defined as the composition of

$$S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\} \to \mathbb{C}P^1, \quad (z_1, z_2) \mapsto [z_1 : z_2].$$

with the diffeomorphism $\Phi : \mathbb{C}P^1 \to S^2$ induced by stereographic projection (see Exercise Sheet 1, Exercise 1). Prove that

$$\pi(z_1, z_2) = \left(2\mathrm{Re}(z_1\overline{z_2}), \quad 2\mathrm{Im}(z_1\overline{z_2}), \quad |z_2|^2 - |z_1|^2\right) \in S^2 \subset \mathbb{R}^3.$$

d) Prove that $H(\pi) = 1$, where π is the Hopf fibration in c).

Solution:

a) If $n = 2k+1$ is odd, then we have that for two regular values y and z, that by the formula of exercise 4 a) of sheet 4

$$H(f) = \ell(f^{-1}(y), f^{-1}(z)) = (-1)^{(2k+1)(2k+1)}\ell(f^{-1}(z), f^{-1}(y)) = -H(f).$$

This implies $H(f) = 0$.

b) Let y, z be regular values for $g \circ f$ and g. Then all points $g^{-1}(y) = \{p_1, \ldots, p_\ell\}$ and $g^{-1}(z) = \{q_1, \ldots, q_k\}$ are also regular values for f. Now we observe that $(g \circ f)^{-1}(y) = \amalg_{i=1}^{\ell} f^{-1}(p_i)$ and $(g \circ f)^{-1}(z) = \amalg_{j=1}^{k} f^{-1}(q_j)$. However the induced orientations, as described in exercise 5 a), on the pieces $f^{-1}(p_i)$ resp $f^{-1}(q_i)$ varies if compared to the one coming from $(g\circ f)^{-1}(y)$ resp. $(g\circ f)^{-1}(z)$ whenever $\mathrm{sign}(dg(p_i)) = -1$ resp. $\mathrm{sign}(dg(q_i)) = -1$. Thus, since the linking number changes by -1 whenever one of the two manifolds has the opposite orientation, we end up exactly with

$$\begin{aligned} H(g\circ f) &= \ell((g\circ f)^{-1}(y),(g\circ f)^{-1}(z)) \\ &= \ell(\amalg_{i=1}^{\ell} f^{-1}(p_i), \amalg_{j=1}^{k} f^{-1}(q_j)) \\ &= \sum_{i=1}^{\ell}\sum_{j=1}^{k} \mathrm{sign}(dg(p_i))\mathrm{sign}(dg(q_i))H(f) \\ &= \deg(g)^2 H(f). \end{aligned}$$

c) Up to ignoring the circle $t \mapsto (e^{it}, 0)$, the diffeomorphism $\Phi : U_1 \to S^2 \setminus \{p_N\} : [z_1 : z_2] \mapsto h^{-1}(\frac{z_1}{z_2})$ where $h^{-1}(z = x+iy) = (\frac{2x}{1+|z^2|}, \frac{2y}{1+|z^2|}, \frac{1-|z|^2}{1+|z|^2})$ is the inverse of the stereographic projection from the North pole. Thus plugging in, we get for $(z_1, z_2) \in S^3$ with $z_2 \neq 0$

$$\begin{aligned} \pi(z_1,z_2) &= h^{-1}\left(\frac{z_1}{z_2}\right) \\ &= \left(\frac{2\mathrm{Re}\left(\frac{z_1}{z_2}\right)}{1+\left|\frac{z_1}{z_2}\right|^2}, \frac{2\mathrm{Im}\left(\frac{z_1}{z_2}\right)}{1+\left|\frac{z_1}{z_2}\right|^2}, \frac{1-\left|\frac{z_1}{z_2}\right|^2}{1+\left|\frac{z_1}{z_2}\right|^2}\right) \\ &= \left(\frac{2|z_2|^2\mathrm{Re}\left(\frac{z_1}{z_2}\right)}{|z_2|^2+|z_1|^2}, \frac{2|z_2|^2\mathrm{Im}\left(\frac{z_1}{z_2}\right)}{|z_2|^2+|z_1|^2}, \frac{|z_2|^2-|z_1|^2}{|z_2|^2+|z_1|^2}\right) \\ &= \left(2\mathrm{Re}(z_1\overline{z_2}),\ 2\mathrm{Im}(z_1\overline{z_2}),\ |z_2|^2-|z_1|^2\right). \end{aligned}$$

This final expression also makes sense for points of the form $(e^{it}, 0)$ and is smooth as a function from $S^3 \to S^2$. So this must also be the expression for those points.

d) Let us take the regular points $y = (0,0,1)$ and $z = (1,0,0)$. Then we get that $A = \pi^{-1}(y) = \{(0,z) : z \in S^1\}$ and $B = \pi^{-1}(z) = \{(\frac{\sqrt{2}}{2}z, \frac{\sqrt{2}}{2}z) : z \in S^1\}$. For the calculation of the Hopf number, we need to use the stereographic projection from a point $p \notin A \cup B$. Here, for convenience let us take $p = (1,0,0,0)$. Then the stereographic projection is given by

$$g : S^3 \setminus \{p\} \to \mathbb{R}^3, \qquad g(z_1 = x_1 + iy_2, z_2) \mapsto \frac{1}{1-x_1}(y_1, z_2).$$

So we have

$$M = g(A) = \{(0,z) \in \mathbb{R} \times \mathbb{R}^2 : z \in S^1\}$$

where we identify $\mathbb{R}^2 = \mathbb{C}$ and

$$N = g(B) = \{(\sin(t),\ \sqrt{2}\cos(t) - 1,\ \sin(t)) \in \mathbb{R}^3 : t \in [0, 2\pi]\}.$$

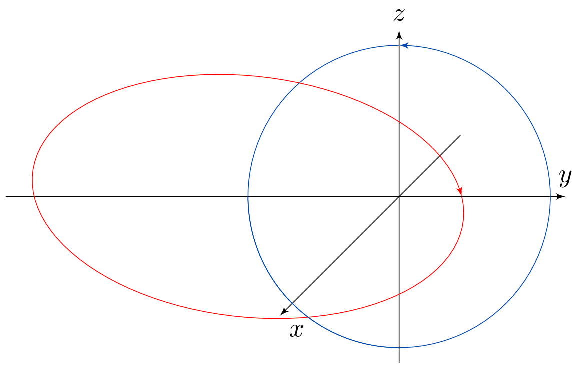

Figure 2: The two linking disks. M is blue and N is red.

By homotopy invariance, as in exercise 5, we can deform N into $N' = \{(z-1, 0) : z \in S^1\}$ in the complement of M.

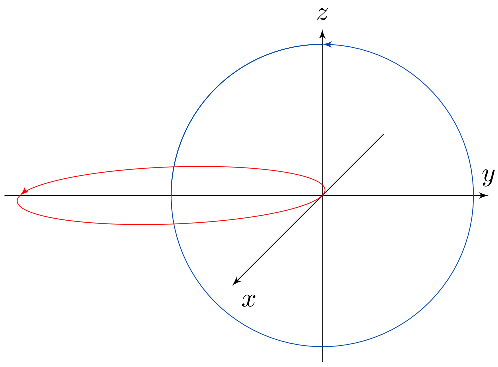

Figure 3: The two linking disks. M is blue and N' is red.

E.g. take $X = \{(\sin(t), (\sqrt{2}(1-s)+s)\cos(t) - 1, s\sin(t)) : s \in [0,1], t \in [0, 2\pi)\}$ is a manifold with boundary $\partial X = N \cup N'$. So by a similar calculation as in Exercise 4,

$$H(\pi) = \ell(N, M) = \ell(N', M) = 1.$$