Solution 6

1. A parallelizable manifold M is one where the tangent bundle TM is isomorphic to the trivial bundle $M \times \mathbb{R}^m$. In other words, M carries m vector fields that are linearly independent at every point of M.

a) Prove that S^3 is parallelizable.

b) Prove that S^7 is parallelizable.

c) Prove that S^{2n} is not parallelizable for all $n \geq 1$.

Hint: For a) and b), see S^3, S^7 as the unit quaternions resp. unit octonions. It is a hard theorem that S^0, S^1, S^3 and S^7 are the only parallelizable spheres.

Solution:

a) We can see S^3 as the set of unit quaternions: Define

$$\mathbb{H} = \{z = x + y\mathbf{i} + v\mathbf{j} + w\mathbf{k} \,:\, (x, y, v, w) \in \mathbb{R}^4\}$$

with relations $\mathbf{ij} = \mathbf{k}$, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and

$$S^3 = \{z = x + y\mathbf{i} + v\mathbf{j} + w\mathbf{k} \in \mathbb{H} \,:\, z\bar{z} = x^2 + y^2 + v^2 + w^2 = 1\}.$$

Here the conjugate is defined as $\bar{z} = x - y\mathbf{i} - v\mathbf{j} - w\mathbf{k}$). We can define three linearly independent vector fields X, Y, Z on S^3 by

$$X(z) = \mathbf{i}z, \ Y(z) = \mathbf{j}z \ \text{ and } Z(z) = \mathbf{k}z.$$

This statement is implied by proving that $\{z, X(z), Y(z), Z(z)\}$ is an orthonormal basis of $\mathbb{R}^4$ for every $z \in S^3$. In the standard basis of $\mathbb{R}^4$ we get

$$M(z) := (z, X(z), Y(z), Z(z)) = \begin{pmatrix} x & -y & -v & -w \\ y & x & w & -v \\ v & -w & x & y \\ w & v & -y & x \end{pmatrix}.$$

Then we see that $M^\top M = \mathbb{1}$ which indeed means that $M \in O(4)$. (As $M(p_N) = \mathbb{1}$, this even gives $SO(4)$.)

Note that the quaternion multiplication gives S^3 the structure of a Lie group and every Lie group is parallelizable. (Simply take a basis of left invariant vector fields. This is exactly what we did by taking X, Y, Z.)

b) We can see S^7 as the set of unit octonions: The octonians are defined as

$$\mathbb{O} = \{z = z_1 + z_2\mathbf{l} \,:\, z_1, z_2 \in \mathbb{H}\}$$

with relations $\mathbf{l}^2 = -1$, $\mathbf{il} = -\mathbf{li}$, $\mathbf{jl} = -\mathbf{lj}$ and $\mathbf{kl} = -\mathbf{lk}$. Then

$$S^7 = \{z = z_1 + z_2\mathbf{l} \in \mathbb{O} \,:\, z\bar{z} = (z_1 + z_2\mathbf{l})(\bar{z}_1 - z_2\mathbf{l}) = |z_1|^2 + |z_2|^2 = 1\}.$$

(Note that $z_2\mathbf{l} = \mathbf{l}\bar{z}_2$.) We can define seven linearly independent vector fields on S^7 by

$$X_1(z) = \mathbf{i}z, \ X_2(z) = \mathbf{j}z, \ X_3(z) = \mathbf{k}z, \ X_4(z) = \mathbf{l}z,$$
$$X_5(z) = \mathbf{il}z, \ X_6(z) = \mathbf{jl}z, \ \text{ and } X_7(z) = \mathbf{kl}z.$$

This statement is implied by proving that

$$\{z, X_1(z).X_2(z), X_3(z), X_4(z), X_5(z), X_6(z), X_7(z)\}$$

is an orthonormal basis for every $z \in S^7$. In the standard basis of $\mathbb{R}^8$ we get

$$M(z) := (z, X_1(z), X_2(z), X_3(z), X_4(z), X_5(z), X_6(z), X_7(z))$$
$$= \begin{pmatrix} x_1 & -y_1 & -v_1 & -w_1 & -x_2 & -y_2 & -v_2 & -w_2 \\ y_1 & x_1 & w_1 & -v_1 & -y_2 & x_2 & w_2 & -v_2 \\ v_1 & -w_1 & x_1 & y_1 & -v_2 & -w_2 & x_2 & y_2 \\ w_1 & v_1 & -y_1 & x_1 & -w_2 & v_2 & -y_2 & x_2 \\ x_2 & y_2 & v_2 & w_2 & x_1 & -y_1 & -v_1 & -w_1 \\ y_2 & -x_2 & w_2 & -v_2 & y_1 & x_1 & -w_1 & v_1 \\ v_2 & -w_2 & -x_2 & y_2 & v_1 & w_1 & x_1 & -y_1 \\ w_2 & v_2 & -y_2 & -x_2 & w_1 & -v_1 & y_1 & x_1 \end{pmatrix}.$$

Then we see that $M^\top M = \mathbb{1}$ which indeed means that $M \in O(8)$. (As $M(p_N) = \mathbb{1}$, this even gives $SO(8)$.)

Note that S^7 with octonian multiplication is not a Lie group since associativity fails.

c) We already know $\chi(S^{2n}) = 2$. So by Poincaré–Hopf, there are no non-vanishing vector fields. Therefore S^{2n} cannot be parallelizable.

2. Let M be a compact manifold.

a) Prove that there exist a finite collection $X_1, X_2, \dots, X_k \in \mathrm{Vect}(M)$ such that $\{X_1(p), X_2(p), \cdots, X_k(p)\}$ spans T_pM at every $p \in M$.

b) Prove that the number k from *a)* can be chosen such that $k \le 2m$, where $m = \dim(M)$.

Hint: For b): Start with any large number of vector fields $X_1, \dots, X_k$ and consider the map $F : T_pM \to \mathbb{R}^k$ whose j-th coordinate is given by $F_j(p, v) := \langle v, X_j(p)\rangle$. What does Sard's theorem tell us when $k > 2m$ and how can you use this to construct $k-1$ vector fields $Y_1, \dots, Y_{k-1}$ which still span T_pM at every $p \in M$?

Solution:

a) We show the following local statement first: For every $p \in M$ exists an open neighborhood $p \in U_p$ and vector field $X_1^{(p)}, \dots, X_m^{(p)} \in \mathrm{Vect}(M)$ such that

$$X_1^{(p)}(q), \dots X_m^{(p)}(q) \text{ is a basis for } T_qM \text{ for every } q \in U_p.$$

Indeed, let $\varphi_p : V_p \to \Omega_p$ be a chart defined on an open neighborhood $p \in V_p \subset M$ with image $\Omega_p \subset \mathbb{R}^m$. Choose a bounded open subset $\varphi_p(p) \in W_p \subset \Omega_p$ with $\overline{W}_p \subset \Omega_p$ and a cut-off function $\rho_p : W_p \to [0, 1]$ with $\rho(x) = 1$ for all $x \in W_p$ and such that $\mathrm{supp}(\rho_p) \subset \Omega_p$ is compact (i.e. ρ_p vanishes near the boundary). Define

$$Y_j : \Omega_p \to \mathbb{R}^m, \qquad Y_j(x) := \rho(x)e_j$$

where $e_1, \dots, e_m$ denotes the standard basis of $\mathbb{R}^m$. We define the vector fields $X_j^{(p)}$ as the pullback of Y_j:

$$X_j^{(p)}(q) := \begin{cases} 0 & q \notin V_p \\ d\varphi(q)^{-1}Y_j(\varphi(q)) & q \in V_p \end{cases}$$

These are clearly smooth vector fields on M and yield a basis of T_qM for all $q \in U_p := \varphi^{-1}(W_p)$.

By compactness of M, there exists finitely many points $p_1, \dots, p_N$ such that $U_{p_1}, \dots, U_{p_N}$ cover M. Then $\{X_j^{(p_i)}\}$ with $j = 1, \dots, m$ and $i = 1, \dots, N$ is a finite collection of vector fields spanning T_pM at every $p \in M$.

b) Let $X_1, \dots, X_k \in \mathrm{Vect}(M)$ be any finite number of vector fields which span T_pM at every point $p \in M$. Choose a Riemannian metric (or embedding $M \subset \mathbb{R}^n$) to define the map

$$F : TM \to \mathbb{R}^k, \qquad F(p,v) = (\langle v, X_1(p)\rangle, \langle v, X_2(p)\rangle, \dots, \langle v, X_k(p)\rangle).$$

Suppose $k > 2m$. Then it follows from Sard's theorem that F is not surjective, since $\dim(TM) = 2m$. Choose $\xi \in \mathbb{R}^k \backslash \mathrm{Image}(F)$. Note that F is homogeneous in the sense that $F(p, tv) = tF(p,v)$ for all $(p,v) \in TM$ and $t \in \mathbb{R}$. This yields the stronger assertion $\mathbb{R}\xi \cap \mathrm{Image}(F) = \{0\}$. We may assume (after relabelling if necessary) that $\xi_k \neq 0$ and define

$$Y_j(p) := \xi_k X_j(p) - \xi_i X_k(p), \qquad \text{for } j = 1, \dots, k-1.$$

We claim that $Y_1, \dots, Y_{k-1}$ span T_pM at every point $p \in M$. If not, then there exists $0 \neq v \in T_pM$ such that

$$0 = \langle Y_j(p), v\rangle = \xi_k \langle X_i(p), v\rangle - \xi_i \langle X_k(p), v\rangle = \xi_k F_i(p,v) - \xi_i F_k(p,v)$$

and hence

$$\begin{aligned} F(p,v) &= \left(\frac{\xi_1}{\xi_k} F_k(p,v), \frac{\xi_2}{\xi_k} F_k(p,v), \dots, \frac{\xi_{k-1}}{\xi_k} F_k(p,v), F_k(p,v) \right) \\ &= \frac{F_k(p,v)}{\xi_k} (\xi_1, \dots, \xi_k). \end{aligned}$$

Since $\mathbb{R}\xi \cap \mathrm{Image}(F) = \{0\}$, it follows $F(p,v) = 0$ and hence $v = 0$. This contradictions our assumption and shows that $Y_1, \dots, Y_{k-1}$ indeed span T_pM.

Repeating the procedure described above, we can decrease the number of vector fields successively until $k \leq 2m$. For $k = 2m$ the argument ceases to work, since F might be surjective. Nevertheless, in many examples one can do better, e.g. for any closed hypersurface $M^m \subset \mathbb{R}^{m+1}$ the coordinate vector fields project to $k = m+1$ vector fields spanning T_pM at every point. Can you find an example where $k = 2m$ vector fields are needed?

3. Let M be a connected manifold without boundary and let $p, q \in M$. Let $K \subset M$ be a compact set containing $p, q \in K^\circ$ in its non empty, connected interior. Then there exists an isotopy $\psi_t : M \to M$ for $t \in [0,1]$ such that $\psi_0 = \mathrm{id}$, $\psi_1(p) = q$ and

$$\mathrm{supp}(\{\psi_t\}_{t\in[0,1]}) := \overline{\bigcup_{0 \leq t \leq 1} \{x \in M : \psi_t(x) \neq x\}} \subset K.$$

We call this an isotopy with compact support in K.

Hint: This is a slight generalization of the homogenisation lemma from the lecture. Have a careful look at the proof and check how it can be adopted to encompass this case.

Solution: Look at the proof of the homogenisation lemma given in class or in Milnor's book at page 22. In this proof, we constructed isotopies F_t supported on a small ball in a chart around any point which sends the origin to any desired point in this ball. Thus say that two interior points $x, y \in K^\circ$ are 'isotopic' in K if there is an isotopy supported in K which sends x to y. By the previous construction in the chart, we see that all points near $y \in K^\circ$ are 'isotopic' to y in K as long as we choose the small ball in the interior of K. "In other words, each isotopy class of points in the interior of K is an open set and the interior of K is partitioned into disjoint open isotopy classes. But the interior of K is connected; hence there can only be one isotopy class." (quote from p:23 of Milnor's book.)

4. Let M^m be a compact, connected manifold without boundary where $m \geq 2$.

a) Let $\varphi : U \to \mathbb{R}^m$ be a chart defined on $U \subset M$ with $B_1(0) \subset \varphi(U)$. Prove that there exists $X \in \mathrm{Vect}(M)$ which does not vanish outside of $\varphi^{-1}(B_1(0))$ and has only isolated non-degenerate zeros in $\varphi^{-1}(B_1(0))$.

b) Let $Y : \overline{B_1(0)} \to \mathbb{R}^m$ be a vector field with isolated non-degenerate zeros. Suppose that $Y(x) \neq 0$ for $x \in \partial B_1(0)$ and $\sum_{x \in Y^{-1}(0)} \iota(Y, p) = 0$. Then there exists a nowhere vanishing vector field $Z : \overline{B_1(0)} \to \mathbb{R}^m$ which agrees with Y near the boundary.

c) Prove that if $\chi(M) = 0$ then there is a vector field $X \in \mathrm{Vect}(M)$ with no zeroes.

Hint: For a): Start with any vector field X with isolated non-degenerate zeros. Use the homogeneity lemma of the previous exercise to move all the zeros into a ball inside a coordinate chart of X.
For b): The Gauss map $\partial B_1(0) \to S^{m-1}$ defined by $x \mapsto Y(x)/\|Y(x)\|$ has degree zero under the given assumptions. Hence it is homotopic to a constant map by the Hopf degree theorem.
For c): Combine a) and b) and make sure that all your modifications produce a smooth vector field.

Solution:

a) Let X be a vector field with isolated non-degenerate zeroes $X^{-1}(0) = \{p_1, \ldots, p_k\}$ and $\varphi : U \to \mathbb{R}^m$ be a chart defined on $U \subset M$ with $B_1(0) \subset \varphi(U)$. Let $\ell = \#X^{-1}(0) \cap \varphi^{-1}(B_1(0))$. If $\ell = k$, we are done. Else pick $q \in \varphi^{-1}(B_1(0))$ with $X(q) \neq 0$, $i \in \{1, \ldots, k\}$ such that $p_i \notin \varphi^{-1}(B_1(0))$ and choose small open neighbourhoods $p_j \in V_j$ for $j \neq i$ such that

$$K := M \setminus (V_1 \cup \cdots \cup V_{j-1} \cup V_{j+1} \cup \cdots \cup V_k)$$

is a compact connected subset with $p_j, q \in K^\circ$. By the previous exercise, there exists an isotoply $\{\psi_t\}$ supported in K with $\psi_1(p_j) = q$. Then $Y = \psi_* X$ is a vector field on M such that $Y^{-1}(0) = \{p_1, \ldots, p_k, q\} \setminus \{p_i\}$. Hence, $\#Y^{-1}(0) \cap \varphi^{-1}(B_1(0)) = \ell + 1$. We can repeat this argument until all the zeroes are in $\varphi^{-1}(B_1(0))$.

b) Since Y has only isolated zeros, it follows from the Hopf lemma that the signed count $\sum_{x \in Y^{-1}(0)} \iota(Y, p)$ agrees with the degree of the Gauss map $f : \partial B_1(0) = S^{m-1} \to S^{m-1}$ defined by $x \mapsto Y(x)/\|Y(x)\|$ (see Exercise 2 on Exercise Sheet 4). Hence this has degree zero and it is homotopic to a constant map by the Hopf degree theorem. Say $H_t : S^{m-1} \to S^{m-1}$ such that $H_1 = f$ and $H_0 \equiv p_N$. Take $\epsilon > 0$ such that $B_1(0) \setminus B_{1-\epsilon}(0) \cap Y^{-1}(0) = \emptyset$. Choose a cut off function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho = 0$ near $1 - \epsilon$, $\rho = 1$ near 1 and ρ is strictly increasing. Define $Z \in \mathrm{Vect}(B_1(0))$ by

$$\begin{cases} Z(p) = p_N & \text{for } p \in B_{1-\epsilon}(0), \\ Z(r\sigma) = (\rho(r)|Y(\sigma)| + (1 - \rho(r)))H_{\rho(r)}(\sigma) & \text{for } r\sigma \in B_1(0) \setminus B_{1-\epsilon}(0), \end{cases}$$

where $r \in [1 - \epsilon, 1)$, $\sigma \in S^{m-1}$. This is a smooth vector field which is non vanishing and agrees with Y near the boundary.

c) Let $\varphi : U \to \mathbb{R}^m$ be a chart defined on $U \subset M$ with $B_1(0) \subset \varphi(U)$. Take as in a) $X \in \mathrm{Vect}(M)$ which does not vanish outside of $\varphi^{-1}(B_1(0))$ and has only isolated non-degenerate zeros in $\varphi^{-1}(B_1(0))$. Then $Y = \varphi_* X|_{\overline{B_1(0)}}$ fulfills the assumptions in b) due to Poincaré–Hopf theorem and $\chi(M) = 0$. Therefore we can find a nowhere vanishing vector field $Z : \overline{B_1(0)} \to \mathbb{R}^m$ which agrees with Y near the boundary. Now define the vector $\tilde{X} \in \mathrm{Vect}(M)$ by

$$\tilde{X}(p) = \begin{cases} X(p), & \text{for } p \notin \varphi^{-1}(B_1(0)), \\ (\varphi^{-1})_* Z, & \text{for } p \in \varphi^{-1}(B_1(0)). \end{cases}$$

By construction, $\tilde{X}$ is smooth and has no zeroes.

5. Let $f : M \to \mathbb{R}$ be a Morse function as in Exercise 2, Sheet 3. The **Morse Lemma** asserts that for every $p \in \mathrm{Crit}(f)$, there is a chart $\varphi_p : U_p \subset M \to \mathbb{R}^m$ with $\varphi_p(p) = 0$ such that

$$f \circ \varphi_p^{-1}(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2.$$

Here $k := \mu(f, p)$ is determined as the dimension of the negative eigenspace of the Hessian $H_p f$ and it is called the **Morse index** of f at p.

Let g be a Riemannian metric on M. The gradient vector field $X = \nabla_g f$ of a smooth function $f : M \to \mathbb{R}$ is defined by

$$g(X(p), \hat{p}) = df(p)\hat{p}, \qquad \text{for all } p \in M \text{ and } \hat{p} \in T_p M.$$

Prove the following properties for the gradient field of a Morse function f.

a) For every metric g the gradiant $X = \nabla_g f$ is a vector field with isolated, non-degenerate zeroes.

b) For every $p \in \mathrm{Crit}(f)$ and any two Riemannian metrics g_1, g_2 the indices $\iota(\nabla_{g_0} f, p) = \iota(\nabla_{g_1} f, p)$ agree.

c) For every $p \in \mathrm{Crit}(f)$ it holds $\iota(\nabla_g f, p) = (-1)^{\mu(f,p)}$.

Solution:

a) The linearization of the gradient vector field ∇f at a critical point p is the linear map

$$\nabla(\nabla f)(p) : T_p M \to T_p M, \qquad \hat{p} \mapsto \nabla_{\hat{p}}(\nabla f)(p).$$

We verify that $\langle \nabla_{\hat{p}}(\nabla f), \cdot \rangle = H_p f(\hat{p}, \cdot)$ is related to the Hessian defined in Exercise 2 of Exercise Sheet 3. Indeed,

$$\begin{aligned} H_p f(X, Y) &= \mathcal{L}_X df(p) Y - df(p) \nabla_X Y \\ &= \mathcal{L}_X \langle \nabla f(p), Y(p) \rangle - \langle \nabla f(p), \nabla_X Y(p) \rangle \\ &= \langle \nabla_X \nabla f(p), Y(p) \rangle. \end{aligned}$$

Here we used that the Levi-Civita connection ∇ is Riemannian. Hence $H_p f$ is nondegenerate if and only if $\nabla(\nabla f)(p)$ is an isomorphism. This shows that every zero is non-degenerate.

Non-degeneracy implies isolated. This follows from the following more general statement. Surjectivity of $\nabla(\nabla f)(p)$ at every point $p \in M$ with $\nabla f(p) = 0$ is equivalent to the statement that $\nabla f : M \to TM$ intersects the zero section $M \times \{0\} \subset TM$ transversely. Therefore the intersection $\{p \in M \mid \nabla f(p) = 0\}$ is a 0-dimensional manifold and hence a discrete set of points.

b) The set of Riemannian metrics is convex. Hence $g_t := (1-t)g_0 + tg_1$ is a Riemannian metric for $0 \leq t \leq 1$. This defines a homotopy of the gradient vector fields defined by $\nabla_{g_t} f$. The zeros for each of these vector field are the same, since $\nabla_{g_t} f(p) = 0$ if and only if $df(p) = 0$ and the later statement does not depend on the metric. It follows from the homotopy invariance of the degree, that all the zeros have the same index for $\nabla_{g_0} f$ and $\nabla_{g_1} f$.

c) Define $h : \mathbb{R}^m \to \mathbb{R}$ by

$$h(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2.$$

Its gradient for the standard metric is the vector field

$$\nabla h(x) : \mathbb{R}^m \to \mathbb{R}^m : \qquad \nabla h(x) := 2(-x_1, \ldots, -x_k, x_{k+1}, \ldots, x_m).$$

Using the Morse Lemma, the index of ∇f at a critical point p of index $k := \mu(f,p)$ agrees with the index of ∇h at the origin. This is defined as degree of the Gauss map $S^{m-1} \to S^{m-1}$ given by

$$(x_1, \ldots, x_m) \mapsto (-x_1, \ldots, -x_k, x_{k+1}, \ldots, x_m).$$

This is a composition of k reflections and hence has degree $(-1)^k$. Hence, it follows $\iota(\nabla_g f, p) = (-1)^{\mu(f,p)}$ for any metric g by part b).

6. **a)** Show that $f_n : \mathbb{R}P^n \to \mathbb{R}$ defined by

$$f_n([x_0 : x_1 : \ldots : x_n]) := \frac{\sum_{j=1}^n j x_j^2}{\sum_{j=0}^n x_j^2}$$

is a Morse function on $\mathbb{R}P^n$. Determine all the critical points of f_n and their Morse index.

b) Show that $\chi(\mathbb{R}P^n) = 0$ when n is odd and $\chi(\mathbb{R}P^n) = 1$ when n is even.

c) Show that $g_n : \mathbb{C}P^n \to \mathbb{R}$ defined by

$$g_n([z_0 : z_1 : \ldots : z_n]) := \frac{\sum_{j=1}^n j|z_j|^2}{\sum_{j=0}^n |z_j|^2}.$$

is a Morse function on $\mathbb{C}P^n$. Determine all the critical points of g_n and their Morse index.

d) Show that $\chi(\mathbb{C}P^n) = n+1$ for $n \geq 1$.

Hint: Use the gradient vector fields of the Morse functions to compute the Euler characteristic in part b) and d) (see Exercise 5 above)

Solution:

a) We write f_n in the standard charts for $\mathbb{R}P^n$. Define $h_i : \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned} h_i(x_1, \ldots, x_n) &:= f_n([x_1 : \cdots : x_{i-1} : 1 : x_i : \cdots : x_n]) \\ &= \frac{\left(\sum_{j=1}^i (j-1)x_j^2\right) + i + \left(\sum_{j=i+1} j x_j^2\right)}{1 + \sum_{j=1}^n x_j^2} \end{aligned}$$

The partial derivatives $\partial_k h_i$ is for $k \leq i$ given by

$$\partial_k h_i(x) = \frac{\left(1 + \sum_{j=1}^n x_j^2\right) 2(k-1)x_j - 2x_k \left[\left(\sum_{j=1}^i (j-1)x_j^2\right) + i + \left(\sum_{j=i+1} j x_j^2\right)\right]}{(1 + \sum_{j=1}^n x_j^2)^2}$$

and for $k > i$ by

$$\partial_k h_i(x) = \frac{\left(1+\sum_{j=1}^n x_j^2\right) 2kx_j - 2x_k\left[\left(\sum_{j=1}^i (j-1)x_j^2\right) + i + \left(\sum_{j=i+1} jx_j^2\right)\right]}{(1+\sum_{j=1}^n x_j^2)^2}.$$

It follows directly that 0 is a critical value of h_i. Inductively, one can show that $\partial_n h_i(x) = 0$ implies $x_n = 0$ and then $\partial_{n-1} h_i(x) = 0$ yields $x_{n-1} = 0$, etc. Therefore 0 is the only critical value of h_i.

When calculating the second partial derivatives, we see that most terms vanish at the origin, and get as final expression

$$\partial_\ell \partial_k h_i(0) = \begin{cases} 2(k-1)\delta_{k\ell} - 2i\delta_{k\ell} & \text{for } k \le i, \\ 2k\delta_{k\ell} - 2i\delta_{k\ell} & \text{for } k > i. \end{cases}$$

Hence $H_0(h_i) = \operatorname{diag}(-2i, -2i+2, \ldots, -2, 2, \ldots, 2n-2j)$. This shows that the origin is a nondegenerate critical point for h_i with Morse index $j-1$.

For the function $f_n : \mathbb{R}P^n \to \mathbb{R}$ it follows that the critical points are

$$p_0 = [1:0:\ldots:0],\ p_1 = [0:1:0:\ldots:0],\ ..., p_n = [0:\ldots:0:1].$$

They are all non-degenerate and have Morse index $\mu(f_n, p_i) = i$.

b) It follows from Exercise 5 and part a) that

$$\chi(\mathbb{R}P^n) = \sum_{i=0}^m (-1)^{\mu(f_n,p_i)} = \sum_{i=0}^m (-1)^i = \begin{cases} 1 & \text{for } n \text{ even}, \\ 0 & \text{for } n \text{ odd}. \end{cases}$$

Note that this is half the Euler characteristic of S^n, which hints at another proof using the Poincaré–Hopf theorem.

c) We proceed similarly to part a). To make things more explicit, we write $z_j = x_j + \mathbf{i}y_j$ and use real notation. In local coordinates the function g_n is represented by $\tilde{h}_i : \mathbb{R}^{2n} \to \mathbb{R}^n$ with

$$\tilde{h}_i(x,y) = \frac{\left(\sum_{j=1}^i (j-1)(x_j^2+y_j^2)\right) + i + \left(\sum_{j=i+1} j(x_j^2+y_j^2)\right)}{1+\sum_{j=1}^n x_j^2+y_j^2}$$

Since $\tilde{h}_i(x,y) = h_i\left(\sqrt{x_1^2+y_1^2}, \ldots, \sqrt{x_n^2+y_n^2}\right)$, it follows from the chain rule and part a) that the only critical point of $\tilde{h}_i$ is the origin. The second partial derivatives are:

$$\partial_{x_k x_k} h_i(0) = \partial_{y_k y_k} h_i(0) = \begin{cases} 2(k-1) - 2i & \text{for } k \le i, \\ 2k - 2i & \text{for } k > i. \end{cases}$$

and all other mixed partial derivative vanish. Hence the Hessian is again a diagonal matrix with precisely $2i$ negative entries.

It follows that $g_n : \mathbb{C}P^n \to \mathbb{R}$ has the critical points

$$p_0 = [1:0:\ldots:0],\ p_1 = [0:1:0:\ldots:0],\ ..., p_n = [0:\ldots:0:1].$$

They are all non-degenerate and have Morse index $\mu(f_n, p_i) = 2i$.

d) It follows from Exercise 5 and part c) that

$$\chi(\mathbb{C}P^n) = \sum_{i=0}^{m} (-1)^{\mu(g_n,p_i)} = \sum_{i=0}^{m} (-1)^{2i} = n+1.$$