Solution 7

1. **a)** For M^{2n-1} compact, oriented manifold without boundary, prove that $\chi(M) = 0$.

b) For $P, Q \subset N$ be two compact oriented submanifolds without boundary of complimentary dimension in an oriented manifold N. Prove that

$$I(Q,P) = (-1)^{\dim(Q)\dim(P)} I(P,Q).$$

c) Let n be odd. Suppose $Q^n \subset N^{2n}$ is a compact oriented submanifold in an oriented manifold N. Show that $I(Q,Q) = 0$.

d) For $N = \mathbb{R}P^2$ and $Q = \mathbb{R}P^1 \subset N$ prove that $I_2(Q,Q) = 1$. Why does this result not contradict c) ?

Hint: For a): Calculate the Euler characteristic using X and $-X$.

Solution:

a) Let $X \in \mathrm{Vect}(M)$ be a vector field with isolated zeros. The Euler characteristic is then given by

$$\chi(M) = \sum_{\{p\colon X(p)=0\}} \iota(X,p).$$

This does not depend on the choice of the vector field X by the Poincaré–Hopf theorem. In particular, $-X \in \mathrm{Vect}(M)$ is another vector field with isolated zeros and therefore

$$\chi(M) = \sum_{\{p\colon X(p)=0\}} \iota(-X,p).$$

We claim that $\iota(-X,p) = -\iota(X,p)$ for every zero p. This implies then $\chi(M) = 0$.

For the claim let $\phi_p : U_p \to \mathbb{R}^{2n-1}$ be a chart around p with $\phi_p(p) = 0$. Then $\iota(X,p)$ is the degree of the map

$$f : S^{2n-2} \to S^{2n-2}, \qquad f(\xi) = \frac{X(\phi^{-1}(\epsilon\xi))}{|X(\phi^{-1}(\epsilon\xi))|}$$

and $\iota(-X,p)$ is the degree of the map

$$g : S^{2n-2} \to S^{2n-2}, \qquad g(\xi) = \frac{-X(\phi^{-1}(\epsilon\xi))}{|X(\phi^{-1}(\epsilon\xi))|}.$$

Note that $g = f \circ \tau$ where $\tau(\xi) = -\xi$ is the antipodal map which has degree $(-1)^{2n-1} = -1$. Therefore

$$\iota(-X,p) = \deg(g) = \deg(f)\cdot\deg(\tau) = -\deg(f) = -\iota(X,p).$$

and this proves the claim.

b) By Thom–Smale transversality we may homotope one of the submanifolds and assume that P and Q intersect transversely. Let $q \in P \cap Q$ be such an intersection point and choose positive bases

$$(v_1,\dots,v_s) \subset T_qP, \qquad (w_1,\dots,w_t) \subset T_qQ.$$

They fit together to form a basis of $T_qP \oplus T_qQ = T_qN$. Define

$$B_1 := (v_1,\dots,v_s; w_1,\dots,w_t), \qquad B_2 := (w_1,\dots,w_t; v_1,\dots,v_s).$$

Then

$$\iota(q; P, Q) = \begin{cases} +1 & B_1 \text{ is a positive basis of } T_q N \\ -1 & B_1 \text{ is a negative basis of } T_q N \end{cases}$$

and

$$\iota(q; Q, P) = \begin{cases} +1 & B_2 \text{ is a positive basis of } T_q N \\ -1 & B_2 \text{ is a negative basis of } T_q N \end{cases}$$

The base change from B_1 to B_2 is given by the matrix

$$T := \begin{pmatrix} 0 & \mathbb{1}_t \\ \mathbb{1}_s & 0 \end{pmatrix}.$$

We can transform T into the identity matrix by st swaps of neighboring columns which shows $\det(T) = (-1)^{st}$ and hence

$$\iota(q; P, Q) = \operatorname{sign}(\det(P))\iota(q; Q, P) = (-1)^{st}\iota(q; Q, P) = (-1)^{\dim(P)\dim(Q)}\iota(q; Q, P).$$

This is valid for all intersection points and hence $I(P, Q) = (-1)^{\dim(P)\dim(Q)} I(Q, P)$.

c) It follows from c) that $I(Q, Q) = -I(Q, Q)$ and therefore $I(Q, Q) = 0$.

d) Identify Q with the submanifold $Q = \{[x_0 : x_1 : 0] \in \mathbb{R}P^2\}$ and define

$$f_t : Q \to \mathbb{R}P^2, \qquad f_t([x_0 : x_1 : 0]) = [\cos(t)x_0 : x_1 : \sin(t)x_0]$$

Then follows $Q' := f_{\pi/2}(Q) = \{[0 : x_1 : x_2] \in \mathbb{R}P^2\}$ and $I_2(Q, Q) = I_2(Q, Q')$ by homotopy invariance of the intersection number. It now follows from the more general calculation in Exercise 5 below, that $I_2(Q, Q') = 1$.

The analog of part b) for mod 2 intersection numbers is the trivial statement $I_2(P, Q) = I_2(Q, P)$. However, this only implies $I_2(Q, Q) = I_2(Q, Q)$ and without signs there is no need for this to vanish.

2. Let M be a compact, oriented manifold without boundary.

a) Let $\Delta = \{(x, x) : x \in M\}$ be the diagonal in $N = M \times M$. Prove that

$$\Delta \cdot \Delta = \chi(M). \tag{1}$$

b) The right-hand side of (1) does not depend on the orientation of M. Does a similar equation to (1) hold for non-oriented manifolds M?

c) Take the zero section $Q = \{(p, v) \in TM : v = 0\}$ in $N = TM$ and prove that

$$Q \cdot Q = \chi(M).$$

Solution:

a) Let $X \in \operatorname{Vect}(M)$ be a vector field with isolated and non-degenerate zeroes. Then we know by the Poincaré–Hopf theorem that

$$\chi(M) = \sum_{p \in M,\, X(p)=0} \iota(p, X).$$

Now we define $Y \in \mathrm{Vect}(\Delta)^\perp$ by $Y(p,p) = (-X(p), X(p))$. This is well-defined, as $T_{(p,p)}\Delta = \{(v,w) \in T_pM \times T_pM \,:\, v = w\}$ and so

$$\langle (v,v), Y(p,p)\rangle = \langle v, X(p)\rangle - \langle v, X(p)\rangle = 0.$$

We also have that p is a zero of X if and only if (p,p) is a zero of Y.

Lets have a look at the orientations in this setup. The orientation of M induces on $N = M \times M$ the product orientation. We oriented the submanifold Δ by decreeing that the map $f : M \to \Delta$ defined by $f(p) = (p,p)$ is an orientation preseving diffeomorphism. With this

$$A_p : T_pM \to T_{(p,p)}\Delta : v \mapsto (v,v)$$

are orientation preserving isomorphism for every $p \in M$. Moreover, consider

$$B_p : T_pM \to (T_{(p,p)}\Delta)^\perp : v \mapsto (-v,v).$$

This is positive in the sense that for every two positive bases $(v_1,\dots,v_n)$ and $(w_1,\dots,w_n)$ of T_pM we have that

$$(A_pv_1,\dots,A_pv_n; B_pw_1,\dots B_pw_n) \subset T_{(p,p)}(M \times M)$$

is a positive basis of $T_{(p,p)}(M \times M)$.

We have that for p a zero of X, the intrinsic derivatives of X and Y are related by

$$D_{(p,p)}Y = B_p \circ D_pX \circ A_p^{-1}. \tag{2}$$

This can be seen by choosing local coordinates: Then

$$X = (X^1,\dots,X^n) : \mathbb{R}^n \to \mathbb{R}^n$$
$$Y = (-X^1,\dots,-X^n, X^1,\dots,X^n) : \mathbb{R}^n \to \mathbb{R}^{2n}$$

and

$$D_pX(v) = (dX^1(p)v,\dots,dX^n(p)v)$$
$$D_pY = (-dX^1(p)v,\dots,-dX^n(p)v, dX^1(p)v,\dots,dX^n(p)v).$$

which proves (2) in local coordinates.

By positivity of A_p and B_p it follows from (2) that $\iota((p,p),Y) = \iota(p,X)$ and both zeros are counted with the same sign. So we may apply the formula for self-intersection numbers in terms of indices of Y as

$$\Delta \cdot \Delta = \sum_{(p,p)\in\Delta,\, Y(p,p)=0} \iota((p,p),Y) = \sum_{p\in M,\, X(p)=0} \iota(p,X) = \chi(M).$$

b) The left hand side can unfortunately not be defined for non-orientable manifolds. What is true though is that

$$I_2(\Delta,\Delta) = \chi(M) \,(\mathrm{mod}\ 2).$$

To prove this take X and Y as in a) and count zeroes, since

$$f : \Delta \to M \times M : (p,p) \to \exp_{(p,p)}(\epsilon Y(p))$$

is transverse to Δ and homotopic to ι_Δ for $\epsilon > 0$ small.

c) We start in the same way as in a). Let $X \in \mathrm{Vect}(M)$ be a vector field with isolated, non-degenerate zeroes. Then we know by Poincaré–Hopf theorem that

$$\chi(M) = \sum_{p \in M, X(p)=0} \iota(p, X).$$

The important observation now is that $f_1 : Q \to TM : (p,0) \mapsto (p, X(p))$ is transverse to the ι_Q and homotopic to ι_Q. Indeed, for the homotopy, simply take $f_t : Q \to TM : (p,0) \mapsto (p, tX(p))$ for $t \in [0,1]$. For transversality, observe that $df_1(p,0)(v,0) = (v, dX(p)v)$. Note that $(p,0) \in f_1^{-1}(Q)$ exactly if $p \in M$, $X(p) = 0$. By the orientation of TM, we know that for $p \in M$ with $X(p) = 0$, the basis $(v_1,0), \dots, (v_m,0), (v_1, dX(p)v_1), \dots, (v_1, dX(p)v_m)$ has sign $\iota(p, X)$. So we get

$$Q \cdot Q = I(f_1, Q) = \sum_{(p,0) \in f_1^{-1}(Q)} \iota(p; f_1, Q) = \sum_{p \in M,\, X(p)=0} \iota(p, X) = \chi(M).$$

3. Let M^m, N^n, Q^{n-m} be compact, oriented manifolds without boundary. Let $f : M \to N$ and $g : Q \to N$ be smooth maps. Explain what it mean that $f \pitchfork g$ and define the intersection number $I(f,g)$. Let $\Delta \subset N \times N$ denote the diagonal. Then

a) $f \pitchfork g$ if and only if $f \times g \pitchfork \triangle$.

b) Show that $I(f,g) = (-1)^{\dim(Q)} I(f \times g, \triangle)$.

Solution: We say that $f \pitchfork g$ if for every triple $p \in M, q \in Q, r \in N$ with $r = f(q) = g(q)$ we have $\operatorname{im} df(p) \oplus \operatorname{im} dg(q) = T_rN$. For every such triple define $\iota((p,q); f, g)$ as the sign of the basis $df(p)v_1, \dots, df(p)v_m, dg(q)w_1, \dots, dg(q)w_{n-m}$ where $(v_i)_i$ is a positive basis of T_pM and $(w_i)_i$ is a positive basis of T_qQ. The intersection number $I(f,g)$ can then be defined as

$$I(f,g) = \sum_{p \in M, q \in Q, f(p)=g(q)} \iota((p,q); f, g)$$

where sum ranges over all these triple. By compactness of M and Q this sum is always finite and this definition agrees with our usual definition if $g = \iota_Q$ is an embedding, in the sense that $I(f, \iota_Q) = I(f, Q)$.

a) For $(p,q) \in M \times Q$ we have $f(p) = g(q)$ if and only if $(f \times g)(p,q) \in \Delta$. Let us investigate what transversality of $f \times g$ to Δ means. We need to be able to write any vector $(u_1, u_2) \in T_{(r,r)}(N \times N)$ as as a sum

$$(u_1, u_2) = (u_3, u_3) + (df(p)v, dg(q)w)$$

for some $u_3 \in T_rN$, $v \in T_pM$, $w \in T_qQ$. Hence, we need to solve $u_1 = u_3 + df(p)v$, and $u_2 = u_3 + dg(q)w$. Subtracting the second one from the first one, yields $u_1 - u_2 = dg(q)w - df(p)v$. If $f \pitchfork g$, then there is $(v,w) \in T_pM \times T_qQ$ such that $u_1 - u_2 = dg(q)w - df(p)v$ holds. Together with $u_3 = u_1 - df(p)v = u_2 - dg(q)w$ we then have found the required components (u_3, v, w). Hence $f \pitchfork g$ implies $f \times g \pitchfork \Delta$.

Assume next $f \times g \pitchfork \Delta$ and take $(p,q) \in M \times Q$ with $f(p) = g(q)$. For $u \in T_rN$ then there exists vectors $(v,w) \in T_pM \times T_qQ$ such that

$$(0, u) = (u', u') + (df(p)v, dg(q)w).$$

So $u' = -df(p)v$ and $u' + dg(q)w = u$. Therefore, $u = df(p)(-v) + dg(q)w$. Since $u \in T_rN$ was arbitrary, it follows $f \pitchfork g$.

b) Let $p \in M$, $q \in Q$ and $r \in N$ with $f(p) = g(q) = r$. We need to understand the relation between $\iota((p,q); f, g)$ and $\iota((r,r); f \times g, \Delta)$. Fix positive bases $(v_i)_i$ of T_pM and $(w_i)_i$ of T_qQ and define

$$B_0 := (u_1, \ldots, u_n) := (df(p)v_1, \ldots, df(p)v_m, dg(q)w_1, \ldots, dg(q)w_{n-m})\,.$$

By transversality $f \pitchfork g$, this is an basis of $T_{f(p)}N$ and

$$\iota((p,q); f, g) = \mathrm{sign}(B_0).$$

On the other hand, let $(\tilde{u}_i)_i$ be a positive basis of T_rN and define

$$B_1 := ((u_1, 0), \ldots, u_m, 0), (0, u_{m+1}), \ldots, (0, u_m), (\tilde{u}_1, \tilde{u}_1), \ldots, (\tilde{u}_n, \tilde{u}_n)).$$

By transversality $f \times g \pitchfork \Delta$, this is a basis of $T_{(r,r)}(N \times N)$ and

$$\iota((p,q); f \times g, \Delta) = \mathrm{sign}(B_1).$$

Since the sign of B_0 is $\iota((p,q); f, g)$, it follows that

$$B_2 := ((u_1, 0), \ldots, u_m, 0), (0, u_{m+1}), \ldots, (0, u_m), (u_1, u_1), \ldots, (u_n, u_n))$$

is a basis with $\mathrm{sign}(B_2) = \iota((p,q); f, g)\mathrm{sign}(B_1)$. Now $(n-m)n$ switches transform B_2 into the basis

$$\begin{aligned} B_3 := ((u_1, 0), \ldots, (u_m, 0), (u_{m+1}, u_{m+1}), \ldots, (u_{n-1}, u_{n-1}), (u_n, u_n) \\ (0, u_{m+1}), \ldots, (0, u_n), (u_1, u_1), \ldots, (u_m, u_m)) \end{aligned}$$

which hence satisfies $\mathrm{sign}(B_3) = (-1)^{n(n-m)}\iota((p,q); f, g)\mathrm{sign}(B_1)$. Further $(n-m)m$ switches yield

$$\begin{aligned} B_4 := ((u_1, 0), \ldots, (u_m, 0), (u_{m+1}, u_{m+1}), \ldots, (u_{n-1}, u_{n-1}), (u_n, u_n), \\ (u_1, u_1), \ldots, (u_m, u_m), (0, u_{m+1}), \ldots, (0, u_n)) \end{aligned}$$

which then satisfies $\mathrm{sign}(B_4) = (-1)^{(n-m)(n+m)}\iota((p,q); f, g)\mathrm{sign}(B_1)$. However, it is not hard to see that this last basis B_4 is positive and hence

$$\iota((p,q); f \times g, \Delta) = \mathrm{sign}(B_1) = (-1)^{(n-m)(n+m)}\iota((p,q); f, g) = (-1)^{(n-m)}\iota((p,q); f, g)$$

we used that $(n-m)$ and $(n+m)$ are either both even or both odd, and hence $(-1)^{(n-m)(n+m)} = (-1)^{(n-m)} = (-1)^{\dim(Q)}$.

If this swapping business is confusing, try to calculate the determinant of the following matrix by Gauss algorithm.

$$\begin{pmatrix} \mathbb{1}_m & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{n-m} \\ \mathbb{1}_m & 0 & \mathbb{1}_m & 0 \\ 0 & \mathbb{1}_{n-m} & 0 & \mathbb{1}_{n-m} \end{pmatrix}$$

4. **a)** Let $f : T^2 \to T^2 : (x, y) \mapsto (ax + by, cx + dy)$ for $a, b, c, d \in \mathbb{Z}$. Prove that the degree and Lefschetz number are given by the formulae

$$\deg(f) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathrm{L}(f) = \det \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}.$$

b) Let $f : S^1 \to S^1$ be a smooth map. Then $\mathrm{L}(f) = 1 - \deg(f)$.

c) Let $f : S^2 \to S^2$ be a smooth map. Then $\mathrm{L}(f) = 1 + \deg(f)$.

Hint: For a), to calculate the degree, set $f_A : T^2 \to T^2 : x \to Ax$. Notice that $\deg(f_{AB}) = \deg(f_A)\deg(f_B)$. With this, you can reduce the general case for $A \neq 0$ to the case $a > 0$ and $\det A \geq 0$. After that, use the identity

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} A \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ ac & a\det A \end{pmatrix}.$$

The Lefschetz Number can be deduced from the degree formula. For b) and c), Hopf degree theorem comes in handily. For c), you can use $\mathbb{C}P^1 \cong S^2$ and use exercise 5 from sheet 4. For maps of degree -1, think antipodal map.

Solution:

a) Degree: We call $f_A : T^2 \to T^2$ the map $(x, y) \to (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$ for $A = (a_{ij}) \in \mathbb{Z}^{2\times 2}$. First of all observe that $f_A \circ f_B = f_{AB}$. We need to prove that $\deg(f_A) = \det(A)$. This works for the null matrix $A = 0$. So assume $A \neq 0$.

Another important observation is that both sides are multiplicative in A.

Claim: We can reduce to the case where $a_{11} > 0$ and $\det A \geq 0$.

Indeed for $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have switch map $f_S : T^2 \to T^2 : (x, y) \mapsto (y, x)$ which has degree (-1) and this agrees with $\det S$. Now one of the matrices SA, A, AS or SAS will have the top left entry non-zero. So it is enough to prove $\deg(f_A) = \det(A)$ for the case $a_{11} \neq 0$.

Another set of special matrices is $T_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. These are products of maps $S^1 \to S^1$ and so $\deg(f_{T_1}) = -1$ and $\deg(f_{T_2}) = -1$ which agrees with $\det(T_1) = \det(T_2) = -1$. Now one of the matrices T_1A, A, AT_2 or T_1AT_2 will have the top left entry positive and its determinant non negative. This proves the claim.

Now we have the identity

$$\begin{pmatrix} 1 & 0 \\ 0 & a_{11} \end{pmatrix} A \begin{pmatrix} 1 & -a_{12} \\ 0 & a_{11} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ a_{21}a_{11} & a_{11}\det A \end{pmatrix}. \tag{3}$$

Calling these matrices A_1, A_2 and A_3, so that the identity rewrites as $A_1AA_2 = A_3$.

$\underline{\deg f_{A_1}}$: 0 is a regular value for f_{A_1}. The system of equations $x = 0, a_{11}y = 0$ has a solutions $p_i = (0, \frac{i}{a})$ for $i = 0, \ldots, a-1$. Each solution contributes $\operatorname{sgn} df_{A_1}(p_i) = \operatorname{sgn} A_1 = \operatorname{sgn} a_{11} = 1$. So $\deg(f_{A_1}) = a_{11}$ and the formula holds.

$\underline{\deg f_{A_2}}$: Also $\deg f_{A_2} = a_{11}$ and the formula holds.

$\underline{\deg f_{A_3}}$: First case is $\det A = 0$. If $a_{21} = 0$, then f_{A_3} cannot hit $(1/2, 1/2)$, so f_{A_3} is not surjective and the formula holds. If $a_{21} = 1$, then f_{A_3} cannot hit $(0, 1/2)$, so f_{A_3} is again not surjective. If $a_{21} \neq 0, 1$, f_{A_3} cannot hit $(1/a_{11}, 1/(a_{11}a_{21}))$, so f_{A_3} is again not surjective.

Finally, if $\det A \neq 0$, 0 is a regular value for f_{A_3}. The system of equations $a_{11}x = 0, a_{21}(a_{11}x) + a_{11}\det(A)y = 0$ is equivalent on T^2 to $a_{11}x = 0, a_{11}\det(A)y = 0$. So we find $a_{11}^2 \det(A)$ solutions that each contribute $\operatorname{sgn}\det A = 1$. This finishes the proof since then (3) reads

$$a_{11}^2 \deg f_A = \deg f_{A_1AA_2} = \deg f_{A_3} = a_{11}^2 \det(A) \Rightarrow \deg f_A = \det A.$$

Lefschetz Number We have that $\mathrm{graph}(f_A)$ is transverse to Δ exactly is A does not have 1 as eigenvalue. Indeed, the following four vectors

$$((1,0), A(1,0)), ((0,1), A(0,1)), ((1,0),(1,0)), ((0,1),(0,1))$$

form a basis, exactly if A does not have 1 as eigenvalue. The sign of this basis is equal to

$$\operatorname{sgn}\det\begin{pmatrix}\mathbb{1} & A\\ \mathbb{1} & \mathbb{1}\end{pmatrix} = \operatorname{sgn}\det\begin{pmatrix}\mathbb{1} & A\\ 0 & \mathbb{1}-A\end{pmatrix} = \operatorname{sgn}\det(\mathbb{1}-A).$$

So every intersection point in the case $\det(\mathbb{1}-A) \neq 0$ contributes $\operatorname{sgn}\det(\mathbb{1}-A)$. How many of these points are there? These are solutions of $f_A(x,y) = (x,z)$ which is equivalent to $f_{\mathbb{1}-A} = 0$. By the degree calculation above, we know that there are exactly $|\det(\mathbb{1}-A)|$ solutions. So indeed,

$$L(f_A) = I(\mathrm{graph}\, f, \Delta) = \det(\mathbb{1}-A).$$

In the case where $\det(\mathbb{1}-A) = 0$, we already know that $\deg(f_{\mathbb{1}-A}) = 0$ and the sign of the differential is constant, we conclude that $\mathbb{1}-A$ cannot be surjective. Take $p \notin f_{\mathbb{1}-A}$, then $f_{\mathbb{1}-A} - c$ does not hit zero, which is equivalent to $f_A - c$ has no fixed point. Thus $L(f_A) = L(f_A - c) = 0$.

b) We know from previous exercises, that f is homotopic to $g(z) = z^k$ where $k := \deg(f)$. Then $\mathrm{L}(f) = \mathrm{L}(g_k)$. First, assume $k > 2$. Then $\mathrm{graph}(g_k)$ is transverse to Δ. The equation $z^k = z$ has $k-1$ solutions on S^1. ($z = 0$ is excluded.) The derivative of g is equal to $g_k'(z) = kz^{k-1} = kz$ for any fixed point, so $dg_k(z)\hat{z} = k\hat{z}$. Therefore, the contribution of any intersection point is equal to the sign of the the basis $(1,k)$, $(1,1)$ of the torus, which has sign $\operatorname{sgn}(1-k)$. Thus $L(g_k) = 1-k$. For $k = 1$, we have identity, so $\mathrm{graph}(g_1) = \Delta$. So by exercise 2, we have $L(g_1) = \Delta\cdot\Delta = \chi(S^1) = 0$. For $k = 0$, $\mathrm{graph}(g_0)$ (a horizontal line) cuts Δ in one point with index 1. So $L(g_0) = 1$. For $k < 0$, write $\ell = -k$. Then the equation $g_k(z) = z^{-\ell} = z \iff z^{\ell+1} = 1$ has $\ell + 1$ solutions on S^1. Each contributes the sign of $(1,-\ell), (1,1)$ which is $\operatorname{sgn}(1+\ell)$. Thus $L(g_k) = 1 + \ell = 1 - k$.

c) We know from Hopf degree theorem, that we only need to consider one map of each degree $k \in Z$. For this we switch perspective and work on $\mathbb{C}P^1$.

$k \geq 2$ For every $k \geq 2$, we already know that $g_k : \mathbb{C}P^1 \to \mathbb{C}P^1 : [z : w] \mapsto [z^k, w^k]$ has degree k. We see that $g_k([0:1]) = [0:1]$. So assume $w = 1$. Then fixed points are solutions of $z^k = z$ which has k solutions. (This time $z = 0$ is also allowed.) So in total, we have $k+1$ fixed points. At every fixed point p of g_k, we have that $(\hat{p}_1, dg_k(p)\hat{p}_1), (\hat{p}_2, dg_k(p)\hat{p}_2)$ is a positive basis whenever $\hat{p}_1, \hat{p}_2$ is positive, since dg_k is complex linear. Alternatively, since g_k is holomorphic, graph(g_k) and Δ are complex submanifolds of $\mathbb{C}P^1 \times \mathbb{C}P^1$ which intersect transversally. So each intersection contributes positively to the intersection number. (Even more generally, pseudo-holomorphic sphere always have positive intersections. Is that not cool? Hope you enjoyed the talk by Dietmar ;)) So in total, we have $L(g_k) = 1 + k$.

$k = 0, 1$ For $k = 1$, we take the identity, and see that $L(g_1) = \Delta \cdot \Delta = \chi(\mathbb{C}P^1) = 2$. For $k = 0$, take the constant map and $L(g_0) = 1$.

$k = -1$ The antipodal map on the sphere has degree -1 and no fixed points, so $L(g_{-1}) = 0$.

$k \leq -2$ For $k < 0$, set $k = -\ell$. Then we take $g_k : \mathbb{C}P^1 \to \mathbb{C}P^1 : [z : w] \mapsto [\bar{z}^\ell, \bar{w}^\ell]$. We still have that $[1 : 0]$ and $[0 : 1]$ are fixed points. So assume $w = 1$, $z \neq 0$. Then we need solutions of $\bar{z}^\ell = z$. $z = 0$ is a solution, so assume $z = \rho e^{i\theta}$ for $\rho > 0$. Then taking the norm of $\bar{z}^\ell = z$ gives $\rho^\ell = \rho$. So this implies $\rho = 1$. However, on S^1, we have that $z^{-\ell} = \bar{z}^\ell = z \Rightarrow z^{\ell+1} = 1$. So we find $\ell + 1$ more fixed points. Since the derivative of g_k vanishes at both $[1 : 0]$ and $[0 : 1]$, these two points contribute positively to $L(g_k)$. All others have non-vanishing complex anti-linear differentials, so they all contribute negatively. In total, we get $L(g_k) = 2 - (1 + \ell) = 1 - \ell = 1 + k$.

5. **a)** For $n < m$ prove that $I_2(\mathbb{R}P^m, \mathbb{R}P^{n-m}) = 1$ in $\mathbb{R}P^n$.

b) Prove that $\mathbb{R}P^n$ is not simply connected.

Solution:

a) We embed $\mathbb{R}P^m$ and $\mathbb{R}P^{n-m}$ into $\mathbb{R}P^n$ as follows

$$\mathbb{R}P^m = \{[x_1 : \cdots : x_m : 0 : \cdots : 0] \in \mathbb{R}P^n\}$$

$$\mathbb{R}P^{n-m} = \{[0 : \cdots : 0 : x_m : \cdots : x_n] \in \mathbb{R}P^n\}.$$

Note that the latter embedding $\mathbb{R}P^{n-m}$ is linearly equivalent (and in particular homotopic) to the standard embedding using the first $(n-m+1)$ coordinates. They intersect in only one point

$$\mathbb{R}P^m \cap \mathbb{R}P^{n-m} = \{[0 : \cdots : 0 : x_m : 0 \cdots : 0] \in \mathbb{R}P^n\}.$$

To see that this intersection is transverse, take the chart

$$\phi_m : U_m \subset \mathbb{R}P^n, \qquad \phi_0([x_0 : \cdots : x_n]) = \left(\frac{x_0}{x_m}, \ldots, \frac{x_{m-1}}{x_m}, \frac{x_{m+1}}{x_m}, \ldots, \frac{x_n}{x_m}\right)$$

where $U_m = \{[x_0 : \cdots : x_n] \,|\, x_m \neq 0\}$. Then

$$\phi_m(\mathbb{R}P^m \cap U_m) = \mathbb{R}^m \times \{0\}, \qquad \phi_m(\mathbb{R}P^{n-m} \cap U_m) = \{0\} \times \mathbb{R}^{n-m}$$

and $\phi_m([0 : \cdots : 0 : x_m : 0 \cdots : 0]) = 0$. Transversality is now obviously satisfied.

Since $\mathbb{R}P^m$ and $\mathbb{R}P^{n-m}$ are transverse and intersect in precisely one point, it follow that their mod 2 intersection index is 1.

b) We claim that $\mathbb{R}P^1 \cong S^1$ is a not contractible loop in $\mathbb{R}P^n$. If $\mathbb{R}P^1$ would be contractible, then it follows that $I_2(\mathbb{R}P^1, \mathbb{R}P^{n-1}) = 0$ by homotopy invariance of the mod 2 intersection number. But we have seen in part (a) that $I_2(\mathbb{R}P^1, \mathbb{R}P^{n-1}) = 1$ and therefore $\mathbb{R}P^1 \subset \mathbb{R}P^n$ is not contractible.

6. Let $f : \mathbb{C} \to \mathbb{C}^2$ be a holomorphic function with coordinates $f(z) := (u(z), v(z))$ and assume that u is non-constant. Let $Q := \{0\} \times \mathbb{C}$.

a) Prove that the intersections of f with Q are all isolated.

b) Prove that the local intersection indices $\iota(z_0; f, Q)$ are all positive.

Solution:

a) We have $f(z) \in Q$ if and only if $u(z) = 0$. Since $u : \mathbb{C} \to \mathbb{C}$ is a holomorphic function, its zeros are all isolated, unless $u \equiv 0$. It follows that all intersections of f with Q are isolated.

b) Suppose $f(z_0) \in Q$. Then $u(z_0) = 0$ and a standard result in complex analysis asserts that

$$u(z) = (z - z_0)^k h(z)$$

for a unique integer $k \geq 1$ and a holomorphic function h with $h(z_0) \neq 0$. Choose a small ball $B_r(z_0) \subset \mathbb{C}$ such that $h(z) \neq 0$ for all $z \in B_r(z_0)$. The intersection index of f with Q at the z_0 is then defined as

$$\iota(z_0; f, Q) := \deg\left(f : S^1 \mapsto S^1,\ f(z) := \frac{u(z_0 + rz)}{|u(z_0 + rz)|}\right).$$

The later map is homotopic to $f_0 = h(z_0)z^k$ by the homotopy

$$f_t(z) := z^k \frac{h(z_0 + trz)}{|h(z_0 + trz)|}$$

and hence $\iota(z_0; f, Q) = k \geq 1$ is positive.

7. Let $Q \subset N$ be a compact submanifold without boundary in a Riemannian manifold N. Prove that there exists a normal vector field $X \in \mathrm{Vect}^{\perp}(Q)$ such that every zero of X is non-degenerate.

Hint: Take a tubular neighbourhood $V_\epsilon := \{\exp_p(v) : q \in Q, v \in (T_qQ)^{\perp}, |v| < \epsilon\}$ of Q. Thom-Smale transversality shows that the exists a embedding map $f_1 : Q \to N$ which is homotopic to the inclusion $f_0 = \iota_Q$ and transverse to Q. Why can we assume that $f_1(Q) \subset V_\epsilon$? (Have a careful look at the proof)

Solution: Let $Q \subset V_\epsilon \subset N$ be a tubular neighborhood of Q. This is similar to Exercise 1 on Exercise sheet 4 where we considered the case of a submanifold in $\mathbb{R}^n$. Let

$$f_0 : Q \hookrightarrow N, \qquad f_0(q) = q$$

be the inclusion map. By Thom–Smale transversality, there exists a map $f_1 : Q \to N$ which is homotopic to f_0 and transverse to Q. Moreover, a careful look at the proof shows that we can assume that f_1 is very close to f_1 in the C^∞-topology. (At some point in the proof we choose a regular value which then yields the function f_1. This regular value can be chosen close to the origin which then yields only a small perturbation) Therefore, we may assume that $f_1(Q) \subset V_\epsilon$. It follows from the properties of the tubular neighborhood that there exists a unique function $\phi : Q \to Q$ and a unique family of normal vectors $Y(q) \in T_{\phi(q)}Q^{\perp}$ with

$$f_1(q) = \exp_{\phi(q)} Y(q).$$

Again, since f_1 can be chosen close to f_0 in the C^∞-topology, we may assume that ϕ is a diffeomorphism. Then $X = \phi^* Y \in \mathrm{Vect}(Q)^\perp$ is a normal vector field along Q and all zeros of X are non-degenerate, because f_1 intersects Q transversely.

8. * Let $Q^n \subset N^{2n}$ be two compact oriented manifolds without boundary and assume that $Q \cdot Q = 0$. Prove that there exists a diffeomorphism $\varphi : N \to N$ isotopic to the identity such that $\varphi(Q) \cap Q = \emptyset$.

Hint: First, construct a normal vector field $X \in \mathrm{Vect}(Q)^\perp$ with no zeros. This uses similar ideas as Exercise 4 on Sheet 6: Start with a vector field X as in Exercise 7, then use the homogeneity Lemma to transport all zeros into a single chart: To transport X along an isotopy $\{\phi_t\}_{0\le t\le 1}$ use parallel transport and define $X_1(q) = \Phi^\perp_{\gamma_q}(1,0)X(q)$ where $\gamma_q : [0,1] \to Q$ is the path $\gamma(t) = \phi_t(q)$. Finally modify the normal vector field in a chart to obtain a nowhere vanishing normal vector field.

Solution: By Exercise 7, there exists a normal vector field $X \in \mathrm{Vect}(Q)^\perp$ with non-degenerated zeros. In particular, X has only finitely many isolated zeros $\{q_1, \dots, q_r\} \subset Q$.

Fix a chart of N around q which is adapted to Q, that is

$$\psi : U \to \mathbb{R}^{2n}, \qquad \psi(U \cap Q) = \psi(U) \cap (\{0\} \times \mathbb{R}^n).$$

We show first that there exits a normal vector field with isolated zeros which are all contained in U and then modify this vector field in the chart.

Let $X \in \mathrm{Vect}(Q)^\perp$ be as above. By the homogeneity Lemma (compare Exercise 3 and 4 of Exercise Sheet 6) there exists an isotopy $\{\phi_t\}_{0\le t\le 1}$ such that $\phi_1(q_1) \in U \cap Q$ and $\phi_t(q_j) = q_j$ for all $j = 2, \dots, r$. Then define

$$X_1(p) = \Phi^\perp_{\gamma_q}(1,0)X(q)$$

where $\gamma_q : [0,1] \to Q$ is the curve $\gamma_q(t) = \phi_t(q)$ and $p = \phi_1(q)$. Then $X_1 \in \mathrm{Vect}(Q)^\perp$ has again only non-degenerated zeros $\{q_1', \dots, q_r\} \subset Q$ with $q_1' \in U \cap Q$.

Now iterate this procedure: Suppose we have a $X_k \in \mathrm{Vect}(Q)^\perp$ with only non-degenerated zeros $\{q_1', \dots, q_k', \dots, q_r\} \subset Q$ with $q_1', \dots, q_k' \in U \cap Q$. Then use the homogeneity Lemma to choose an isotopy $\{\phi_t\}_{0\le t\le 1}$ such that $\phi_1(q_{k+1}) \in K$ and $\phi_t(q_j) = q_j$ for all $j \neq k$. Define as before

$$X_{k+1}(p) = \Phi^\perp_{\gamma_q}(1,0)X_k(q)$$

where $\gamma_q : [0,1] \to Q$ is the curve $\gamma_q(t) = \phi_t(q)$ and $p = \phi_1(q)$ for this new isotopy. Then X_{k+1} has again only non-degenerated zeros $\{q_1', \dots, q_{k+1}', \dots, q_r\} \subset Q$ with $q_1', \dots, q_{k+1}' \in U \cap Q$.

We end up with $X_r \in \mathrm{Vect}(Q)^\perp$ which has only isolated zeros $\{q_1', \dots, q_r'\} \subset U \cap Q$. Consider its push-forward

$$Y : \mathbb{R}^n \to \mathbb{R}^n \cong \mathbb{R}^n \times \{0\}, \qquad Y(\xi) = d\psi(\psi^{-1}(\xi))X_r(\psi^{-1}(\xi)).$$

The index of a zero q_k' of X_r is ± 1 where the sign is determined by the sign of the determinant of $dY(\psi(q_k')$. It follows

$$0 = Q \cdot Q = \sum_{k=1}^r \iota(X_r, q_k') = \sum_{\xi \in Y^{-1}(0)} \iota(Y, \xi).$$

Note that we can view Y as an honest vector field on $\mathbb{R}^n$. This has only finitely many zeros and their signed count vanishes. We can now apply Exercise 4 from Exercise Sheet

7 to obtain a vector field $\tilde{Y} : \mathbb{R}^n \to \mathbb{R}^n$ which has no zeros and agrees with Y outside of a compact region in $\mathbb{R}^n$. Use this to define $\tilde{X} \in \mathrm{Vect}(Q)^\perp$ by

$$\tilde{X}(q) := \begin{cases} X_k(q) & \text{for } q \in Q \backslash U \\ d\psi(q)^{-1}(\tilde{Y}(\psi(q)), 0) & \text{for } q \in U \cap Q \end{cases}$$

This is then a normal vector field along Q which has no zeros.

Finally, extend $\tilde{X} \in \mathrm{Vect}(Q)^\perp$ to a smooth vector field $\hat{X} \in \mathrm{Vect}(N)$ and denote by φ_t the flow of $\hat{X}$. Then follows $\varphi_t(Q) \cap Q = \emptyset$ for all $0 < t < \epsilon$ for sufficiently small ϵ.