Solution 8

1. **a)** Let $\dim(V) = n$ and let $\alpha_i \in V^*$, $v_i \in V$ for $i = 1, \dots, n$. Prove that

$$(\alpha_1 \wedge \ldots \wedge \alpha_n)(v_1, \dots, v_n) = \det(\alpha_j(v_j)_{ij}).$$

b) Prove that $dx^I = dx^{i_1} \wedge \ldots dx^{i_k}$ for $I = (i_1, \dots, i_k) \in \mathcal{I}_k$.

c) Let $\dim(V) = n$. Prove that for a linear map $\Phi : V \to V$ and $\omega \in \Lambda^n V^*$, we have

$$\Phi^* \omega = \det(\Phi)\omega.$$

Solution:

a) We prove this result by induction on the dimension m. The case $m = 1$ is clear. For $m > 1$, we have

$$\begin{aligned}
&(\alpha_1 \wedge (\alpha_2 \wedge \ldots \wedge \alpha_m))(v_1, \dots, v_m) \\
&:= \sum_{i=1}^{m} (-1)^{i-1} \alpha_1(v_i)(\alpha_2 \wedge \ldots \wedge \alpha_m)(v_1, \dots, \hat{v}_i, \dots, v_m) \\
&= \sum_{i=1}^{m} (-1)^{i-1} a_{1i} M_{1i}
\end{aligned}$$

where $\operatorname{sgn}(v_i, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_m) = (-1)^{i-1}$, $A = (a_{ij})_{i,j=1}^m = (\alpha_i(v_j))_{i,j=1}^m$ and M_{1i} is equal to $(\alpha_2 \wedge \ldots \wedge \alpha_m)(v_1, \dots, \hat{v}_i, \dots, v_m)$. By induction, M_{1i} is equal to the determinant of A with the i-th column and the first line deleted. Hence by Laplace development, we have

$$(\alpha_1 \wedge (\alpha_2 \wedge \ldots \wedge \alpha_m))(v_1, \dots, v_m) = \sum_{i=1}^{m} (-1)^{i-1} a_{1i} M_{1i} = \det(A).$$

b)

$$dx^I(v_1, \dots, v_k) = \det \begin{pmatrix} v_1^{i_1} & \cdots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \cdots & v_k^{i_k} \end{pmatrix} \overset{a)}{=} dx^{i_1} \wedge \ldots \wedge dx^{i_m}(v_1, \dots, v_k).$$

c) By linearity, we only need to look at elements $\omega = dx^I$ where $I = (1, 2, \dots, n)$.

$$\begin{aligned}
\Phi^* dx^I(v_1, \dots, v_k) &:= dx^I(\Phi v_1, \dots, \Phi v_n) = \det(\Phi v_1, \dots, \Phi v_n) \\
&= \det(\Phi(v_1, \dots, v_n)) = \det(\Phi) \det(v_1, \dots, v_n) \\
&= \det(\Phi) dx^I(v_1, \dots, v_n).
\end{aligned}$$

2. We look at the exterior differential on $\mathbb{R}^3$.

a) For $f \in \Omega^0(\mathbb{R}^3) = C^\infty(\mathbb{R}, \mathbb{R}^3)$, prove that

$$df = \sum_{i=1}^{3} \mathbf{grad}(f)_i \, dx^i,$$

where $\mathbf{grad}(f) = (\partial_1 f, \partial_2 f, \partial_3 f)$.

b) For $\alpha \in \Omega^1(\mathbb{R}^3)$ with $\alpha = g_1 dx^1 + g_2 dx^2 + g_3 dx^3$, prove that

$$d\alpha = \mathbf{curl}(g)_1\, dx^2 \wedge dx^3 + \mathbf{curl}(g)_2\, dx^3 \wedge dx^1 + \mathbf{curl}(g)_3\, dx^1 \wedge dx^2,$$

where $\mathbf{curl}(g) = (\partial_2 g_3 - \partial_3 g_2, \partial_3 g_1 - \partial_1 g_3, \partial_1 g_2 - \partial_2 g_1)$ for $g \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

c) For $\omega \in \Omega^2(\mathbb{R}^3)$ with $\omega = h_1 dx^2 \wedge dx^3 + h_2 dx^3 \wedge dx^1 + h_3 dx^1 \wedge dx^2$, prove that

$$d\omega = \mathbf{div}(h)\, dx^1 \wedge dx^2 \wedge dx^3,$$

where $\mathbf{div}(h) = \partial_1 h_1 + \partial_2 h_2 + \partial_3 h_3$ for $h \in C^\infty(\mathbb{R}^3, \mathbb{R}^3)$.

d) Prove that $\mathbf{curl} \circ \mathbf{grad} = 0$ and $\mathbf{div} \circ \mathbf{curl} = 0$.

So we have

$$\Omega^0(\mathbb{R}^3) \xrightarrow{d=grad} \Omega^1(\mathbb{R}^3) \xrightarrow{d=cur\ell} \Omega^2(\mathbb{R}^3) \xrightarrow{d=div} \Omega^3(\mathbb{R}^3).$$

Solution:

a) By definition, $df = \sum_{i=1}^3 \partial_i f_i dx^i = \sum_{i=1}^3 \mathbf{grad}(f)_i\, dx^i$.

b) By definition,

$$\begin{aligned} d\alpha &= \sum_{j,i=1}^{3} \partial_i g_j dx^i \wedge dx^j = \partial_2 g_3 dx^2 \wedge dx^3 + \partial_3 g_2 dx^3 \wedge dx^2 + \dots \\ &= (\partial_2 g_3 - \partial_3 g_2) dx^2 \wedge dx^3 + (\partial_3 g_1 - \partial_1 g_3) dx^3 \wedge dx^1 + (\partial_1 g_2 - \partial_2 g_1) dx^1 \wedge dx^2 \\ &= \mathbf{curl}(g)_1\, dx^2 \wedge dx^3 + \mathbf{curl}(g)_2\, dx^3 \wedge dx^1 + \mathbf{curl}(g)_3\, dx^1 \wedge dx^2. \end{aligned}$$

c) We have

$$\begin{aligned} d\omega &= \partial_1 h_1 dx^1 \wedge dx^2 \wedge dx^3 + \partial_2 h_2 dx^2 \wedge dx^3 \wedge dx^1 + \partial_3 h_3 dx^3 \wedge dx^1 \wedge dx^2 \\ &= \mathbf{div}(h)\, dx^1 \wedge dx^2 \wedge dx^3. \end{aligned}$$

d) This follows from $d^2 = 0$.

3. Prove that a compact manifold M^m is orientable if and only if there exists a nowhere vanishing m-form ω on M. Such a form is called **volume form**.

Solution: Let us start with an M^m manifold that is oriented. Take a cover by positive charts $\varphi_i : U_i \to \mathbb{R}^n$ for $i = 1, \dots, N$. Take a partition of unity $\rho_i : M \to [0,1]$ subordinate to the cover $\{U_i\}_i = 1^N$. Then we define forms $\omega_i \in \Omega^m(M)$ by

$$(\omega_i)_p = \begin{cases} \rho_i(p)(\varphi_i)^*(dx^1 \wedge \ldots \wedge dx^m)_p & \text{for } p \in \operatorname{supp} \rho_i, \\ 0 & \text{for } p \notin \operatorname{supp} \rho_i. \end{cases}$$

where $dx^1 \wedge \ldots \wedge dx^m$ is the standard volume form on $\mathbb{R}^m$. Put $\omega = \sum_{i=1}^N \omega_i$. We will now prove that this form is nowhere vanishing. Take $p \in U_i$ for some $i \in \{1, \dots, N\}$. Then we can push ω into the chart φ_i. For $x \in \varphi_i(U_i)$,

$$\begin{aligned} ((\varphi_i^{-1})^*\omega)_x &= \sum_{j=1}^N \rho_j(\varphi_i^{-1}(x))(\varphi_i^{-1})^*\omega_j)_x = \sum_{j=1}^N \rho_j(\varphi_i^{-1}(x))(\varphi_i^{-1})^*(\varphi_j)^*(dx^1 \wedge \ldots \wedge dx^m))_x \\ &= \left(\sum_{j=1}^N \rho_j(\varphi_i^{-1}(x)) \det(d(\varphi_i^{-1} \circ \varphi_j)(\varphi_j \circ \varphi_i^{-1}(x))) \right) dx^1 \wedge \ldots \wedge dx^m, \end{aligned}$$

where we used Exercise 1 in the last equality. To see that this coefficient is non-zero, we notice that $\det(d(\varphi_i^{-1}\circ\varphi_j)(\varphi_j\circ\varphi_i^{-1}(x)))>0$ for all $i,j=1,\dots,N$, because all the charts were positive. Also since $\sum_{j=1}^N \rho_i \equiv 1$, at p at least one $\rho_k(p)>0$ and all others are ≥ 0. So in total, we have for $x=\varphi_i(p)$

$$\sum_{j=1}^N \rho_j(\varphi_i^{-1}(x))\det(d(\varphi_i^{-1}\circ\varphi_j)(\varphi_j\circ\varphi_i^{-1}(x)))$$
$$=\underbrace{\rho_k(p)\det(d(\varphi_i^{-1}\circ\varphi_k)(\varphi_k(p)))}_{>0}+\underbrace{\sum_{j\neq k,j=1}^N \rho_j(p)\det(d(\varphi_i^{-1}\circ\varphi_j)(\varphi_j(p)))}_{\geq 0}>0.$$

So ω does not vanish at $p\in U_i$, but since this was an arbitrary choice, ω is nonzero for any $p\in M$. So ω is a volume form.

Now let us assume we have $\omega\in\Omega^m(M)$ a volume form. Then we can define an orientation, by saying that a basis $e_1,\dots,e_m\in T_pM$ is positive if $\omega(e_1,\dots,e_m)>0$. These orientation fit together smoothly. Namely, fix a point $p_0\in M$ and a positive basis $e_1,\dots,e_m$ of $T_{p_0}M$ and choose vector fields $X_1,\dots,X_m\in\mathrm{Vect}(M)$ such that $X_i(p_0)=e_i$ for $i=1,\dots,m$. Then there is a connected open neighborhood $U\subset M$ of p_0 such that the vectors $X_1(p),\dots,X_m(p)$ form a basis of T_pM for every $p\in U$. Hence the function

$$U\to\mathbb{R}:p\mapsto\omega_p(X_1(p),\dots,X_m(p))$$

is everywhere nonzero and hence is everywhere positive, because it is positive at $p=p_0$. Thus the vectors $X_1(p),\dots,X_m(p)$ form a positive basis of T_pM for every $p\in U$.

4. Let V be a m-dimensional real vector space and $\omega\in\Lambda^kV^*\setminus\{0\}$ be an alternating k-form on V. The kernel of ω is the linear subspace

$$\ker(\omega):=\{v\in V\,:\,\omega(v,v_2,\dots,v_k)=0\text{ for all }v_2,\dots,v_k\in V\}.$$

a) Prove that $\dim(\ker(\omega))\leq m-k$.

b) We call $\omega\in\Lambda^kV^*$ *decomposable* if there are $\alpha_i\in V^*$ such that

$$\omega=\alpha_1\wedge\ldots\wedge\alpha_k.$$

Show that ω is decomposable if and only if $\dim(\ker(\omega))=m-k$.

c) Show that for $\dim(V)<4$, every k-form is decomposable.

d) We call $\omega\in\Lambda^kV^*$ *non-degenerate* if $\ker(\omega)=0$. Find an example of a non-degenerate 2-form on $\mathbb{R}^4$.

Solution:

a) Suppose $\dim(\ker(\omega))>m-k$ and let $v_1,\dots,v_k\in V$, we claim that

$$\omega(v_1,\dots,v_k)=0. \tag{1}$$

Suppose first $v_1,\dots,v_k$ are not linearly independent. Then there exist $\lambda_i\in\mathbb{R}$ with $\sum_{i=1}^n\alpha_iv_i=0$ and we assume without loss of generality $\lambda_1\neq0$. Then follows from the properties of an alternating form that

$$\omega(v_1,\dots,v_k)=\frac{1}{\lambda_1}\omega\left(\sum_{i=1}^n\alpha_iv_i,v_2,\dots,v_n\right)=\frac{1}{\lambda_1}\omega(0,v_2,\dots,v_n)=0$$

and this proves (1) in this case.

Otherwise suppose $v_1, \ldots, v_k$ are linearly independent. Then follows from dimension reasons that the span of $v_1, \ldots, v_k$ must intersect $\ker(\omega)$. Hence there exists $\lambda_i \in \mathbb{R}$ with $\sum_{i=1}^n \alpha_i v_i \in \ker(\omega)$ and we may assume without loss of generality $\lambda_1 \neq 0$. Then follows from the properties of an alternating form that

$$\omega(v_1, \ldots, v_k) = \frac{1}{\lambda_1}\omega\left(\sum_{i=1}^n \alpha_i v_i, v_2, \ldots, v_n\right) = 0.$$

and this proves (1) in the second case.

Hence (1) holds for all vectors $v_1, \ldots, v_k \in V$ and thus $\omega = 0$.

b) Suppose ω is decomposable and

$$\omega = \alpha_1 \wedge \ldots \wedge \alpha_k.$$

for some $\alpha_i \in V^*$. Observe first that the α_i must be linearly independent in V^* when $\omega \neq 0$. We may thus extend them to a basis $\alpha_1, \ldots, \alpha_n$ on V^*. Let $u_1, \ldots, u_n$ be the dual basis in V. Then it is not hard to see that $u_{k+1}, \ldots, u_n \in \ker(\omega)$ and hence $\dim\ker(\omega) \geq (n-k)$. It follows from part (a) that equality holds.

Conversely, suppose $\dim\ker(\omega) = (n-k)$. Choose a basis of $u_1, \ldots, u_n$ of V such that $u_{k+1}, \ldots, u_k$ span $\ker(\omega)$. Let $\alpha_1, \ldots, \alpha_n \in V^*$ be the dual basis. We know that the forms

$$\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}, \qquad \text{with } 1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

form a basis of $\Lambda^k V^*$. Hence there exist coefficients $\lambda_{i_1,\ldots,i_k} \in \mathbb{R}$ such that

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1,\ldots,i_k} \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}.$$

We can recover the coefficients by the relation

$$\lambda_{i_1,\ldots,i_k} = \omega(u_{i_1}, \ldots, u_{i_k}).$$

And since $u_{k+1}, \ldots, u_n$ are all in the kernel of ω, it follows that all coefficients vanish except the one corresponding to $u_1, \ldots, u_n$, i.e.

$$\omega = \lambda_{1,\ldots,k} \alpha_1 \wedge \cdots \wedge \alpha_k$$

and ω is decomposable.

c) Every 1-form is decomposable. If $\dim(V) = 2$, then $\Lambda^2 V^*$ is one-dimensional and spanned by $\alpha_1 \wedge \alpha_2$ for any basis α_1, α_2 of V^*.

If $\dim(V) = 3$, then $\Lambda^3 V^*$ is one-dimensional and spanned by $\alpha_1 \wedge \alpha_2 \wedge \alpha_3$ for any basis $\alpha_1, \alpha_2, \alpha_3$ of V^*.

The nontrivial part is to show that every 2-form on a three-dimensional space is decomposable. By part b) this is equivalent to showing that every three form has a kernel. Let u_1, u_2 be two vectors with $\omega(u_1, u_2) \neq 0$ and define

$$W := \{v \in V \,|\, \omega(u_1, v) = \omega(u_2, v) = 0\}$$

Since $\{v \in V : \omega(u_i, v) = 0\}$ are hyperplanes in V, it follows $\dim(W) \geq 1$. And since the span of u_1, u_2 intersects V only in the origin, it follows

$$V = W \oplus \operatorname{span}(u_1, u_2)$$

It is now easy to see that $W = \ker(\omega)$: Let $w \in W$. For $v \in \operatorname{span}(u_1, u_2)$, it follows from the definition of W that $\omega(w, v) = 0$. For $v \in W$, we must have $v = \lambda w$, since $\dim(W) = 1$, and hence $\omega(w, v) = \lambda\omega(w, w) = 0$.

d) The standard example of a non-degenerate 2-form on $\mathbb{R}^4$ is

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4.$$

One readily checks that $\ker(\omega) = 0$.

5. The **Hodge star** operator $* : \Omega^k(\mathbb{R}^m) \to \Omega^{m-k}(\mathbb{R}^m)$ is the unique map satisfying

$$\eta \wedge *\omega = \langle \omega, \eta \rangle dx^1 \wedge \cdots \wedge dx^m$$

for all $\omega, \eta \in \Omega^k(\mathbb{R}^m)$. Here the inner product of two k-forms is defined by $\left\langle \sum_{I \in \mathcal{I}_k} a_I dx^I, \sum_{I \in \mathcal{I}_k} b_I dx^I \right\rangle = \sum_{I \in \mathcal{I}_k} a_I b_I$.

a) Calculate $*\omega \in \Omega^1(\mathbb{R}^3)$ for

$$\omega = a_{12} dx^1 \wedge dx^2 + a_{13} dx^1 \wedge dx^3 + a_{23} dx^2 \wedge dx^3 \in \Omega^2(\mathbb{R}^3)$$

b) Show that $* : \Omega^k(\mathbb{R}^m) \to \Omega^{m-k}(\mathbb{R}^m)$ is linear.

c) Calculate all $\omega \in \Omega^2(\mathbb{R}^4)$ with $\omega = *\omega$.

d) Calculate $** \omega$ for a general form $\omega \in \Omega^k(\mathbb{R}^m)$.

Solution:

a) We have

$$*\omega = a_{12} dx^3 - a_{13} dx^2 + a_{23} dx_1$$

as one verifies directly for

$$\eta = b_{12} dx^3 - b_{13} dx^2 + b_{23} dx_1$$

that $\omega \wedge *\eta = (a_{12} b_{12}^2 + a_{13} b_{13} + a_{23} b_{23}) dx^1 \wedge dx^2 \wedge dx^3$.

b) This can be most easily shown by deriving an explicit formula for $*\omega$. Let us introduce some notation: Denote by $\mathcal{I}_k$ the collection of all tuples $I = (i_1, \ldots, i_k)$ with $1 \leq i_1 < \cdots < i_k \leq m$ and define

$$dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Moreover, denote by $\bar{I} = (j_1, \ldots, j_{m-k})$ the complementary tuple with $1 \leq j_1 < \cdots < j_{m-k} \leq m$ such that $\{i_1, \ldots, i_k, j_1, \ldots, j_{m-1}\} = \{1, \ldots, n\}$ and define $\epsilon(I) \in \{\pm 1\}$ by

$$\epsilon(I) dx^I \wedge dx^{\bar{I}} = dx^1 \wedge \cdots \wedge dx^m$$

We then have that

$$*\left(\sum_{I \in \mathcal{I}_k} a_I dx^I\right) = \sum_{I \in \mathcal{I}_k} \epsilon(I) a_I dx^{\bar{I}}.$$

and this is clearly linear.

c) The following three 2-forms satisfy $\omega = *\omega$:

$$dx^1 \wedge dx^2 + dx^3 \wedge dx^4, \qquad dx^1 \wedge dx^3 - dx^2 \wedge dx^4, \qquad dx^1 \wedge dx^4 + dx^2 \wedge dx^3.$$

The following three 3-forms satisfy $\omega = - * \omega$:

$$dx^1 \wedge dx^2 - dx^3 \wedge dx^4, \qquad dx^1 \wedge dx^3 + dx^2 \wedge dx^4, \qquad dx^1 \wedge dx^4 - dx^2 \wedge dx^3.$$

The space of 2-forms in $\mathbb{R}^4$ is 6-dimensional and it is not hard to see that the six forms above form a basis. A 2-form ω is self-dual, i.e. satisfies $\omega = *\omega$, if and only if it is in the span of the first three forms.

d) It follows from the formula in part b) that

$$**\left(\sum_{I \in \mathcal{I}_k} a_I dx^I\right) = \sum_{I \in \mathcal{I}_k} \epsilon(I)\epsilon(I') a_I dx^{\bar{I}}.$$

These signs $\epsilon(I)$ and $\epsilon(I')$ satisfy

$$\epsilon(I) dx^I \wedge dx^{\bar{I}} = dx^1 \wedge \cdots \wedge dx^m = \epsilon(\bar{I}) dx^{\bar{I}} \wedge dx^I.$$

Moreover, super commutativity of the wedge product shows

$$dx^I \wedge dx^{\bar{I}} = (-1)^{m(m-k)} dx^{\bar{I}} \wedge dx^I$$

Combining both expressions yields $\epsilon(I)\epsilon(I') = (-1)^{m(m-k)}$ and hence $**\omega = (-1)^{m(m-k)}\omega$.

6. **a)** Define $\omega \in \Omega^2(S^2)$ by

$$\omega_x = x_1 dx^2 \wedge dx^3 + x_2 dx^3 \wedge dx^1 + x_3 dx^1 \wedge dx^2.$$

Use Stokes' theorem to establish a relation between $\text{area}(S^2) := \int_{S^2} \omega$ and $\text{vol}(B_1(0))$. Calculate either one of them, to deduce the other.

b) Define $\rho \in \Omega^n(S^n)$ by

$$\rho_x := \sum_{i=1}^{n+1} (-1)^{j-1} x_j dx^1 \dots dx^{j-1} \wedge \widehat{dx^j} \wedge dx^{j+1} \dots dx^{n+1},$$

This is obtained by plugging x into the first coordinate of the standard volume form on $\mathbb{R}^{n+1}$. Prove that ρ is invariant under $\text{SO}(n+1)$ and that ρ is a volume form on S^n.

Solution:

a) We have

$$d\omega = 3dx^1 \wedge dx^2 \wedge dx^3$$

and hence by Stokes theorem

$$\text{area}(S^2) = \int_{S^2} \omega = \int_{B_1(0)} 3dx^1 \wedge dx^2 \wedge dx^3 = \int_{B_1(0)} 3dx_1 dx_2 dx_3 = 3\text{vol}(B_1(0))$$

where $B_1(0) \subset \mathbb{R}^3$ denotes the unit ball. We calculate the last intgral with Fubini:

$$\text{vol}(B_1(0)) = \int_{-1}^{1} \text{area}\{x^2 + y^2 \leq 1 - z^2\}\, dz = \int_{1}^{1} \pi(1 - z^2) = \frac{4}{3}\pi$$

It then follows $\text{area}(S^2) = 4\pi$.

b) Denote the standard volume form on $\mathbb{R}^{n+1}$ by

$$\mathrm{dvol}_{\mathbb{R}^{n+1}} := dx^1 \wedge \cdots \wedge dx^{n+1}.$$

This satisfies for any $A \in \mathbb{R}^{(n+1)\times(n+1)}$ the relation

$$A^*\mathrm{dvol}_{\mathbb{R}^{n+1}} := \det(A) dx^1 \wedge \cdots \wedge dx^{n+1}$$

For $A \in \mathrm{SO}(n)$ it hence follows $A^*\mathrm{dvol}_{\mathbb{R}^{n+1}} = \mathrm{dvol}_{\mathbb{R}^{n+1}}$. It follows

$$\begin{aligned}(A^*\rho)_x(v_1, \ldots, v_n) &= \rho_{Ax}(Av_1, \ldots, Av_n) \\ &= \mathrm{dvol}_{\mathbb{R}^{n+1}}(Ax, Av_1, \ldots, Av_n) \\ &= \mathrm{dvol}_{\mathbb{R}^{n+1}}(x, v_1, \ldots, v_n) \\ &= \rho_x(v_1, \ldots, v_n)\end{aligned}$$

and hence $A^*\rho = \rho$.

Next, let $x \in S^n$ and choose a basis $v_2, \ldots, v_{n+1}$ of $T_xS^n = (\mathbb{R}x)^\perp$. Then follows that $x, v_2, \ldots, v_{n+1}$ is a basis of $\mathbb{R}^{n+1}$ and hence

$$0 \neq \det(x, v_2, \ldots, v_{n+1}) = \mathrm{dvol}_{\mathbb{R}^{n+1}}(x, v_2, \ldots, v_{n+1}) = \rho_x(v_2, \ldots, v_{n+1}).$$

This shows that $\rho_x \neq 0$ for all $x \in S^n$ and so ρ is a volume form.